Fuzzy ideals of Dual QS-algebras

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Abstract:
The aim of this paper is to introduce the notion of fuzzy KUS-ideal in a KUS-algebra, several theorems, properties are stated and proved. The fuzzy relations on KUS-algebras are also studied.

Keywords: KUS-algebra, fuzzy KUS-sub-algebra, fuzzy KUS-ideal, homomorphisms of KUS-algebras, image and pre-image of fuzzy KUS-ideals.

1. Introduction
The concept of fuzzy subset was introduced by L.A. Zadeh in [6], and was used afterwards by many authors in various branches of mathematics. Particularly in the area of fuzzy topology. Much research has been carried out since. In 1966 [9], Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [4],[5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Xi [7] applied the concept of fuzzy subset to BCK-algebras and gave some of its properties. J. Neggers, S.S. Ahn and H.S. Kim [3] introduced a Q-algebra, which is a generalization of BCI/BCK-algebras and generalized some theorems discussed in BCI-algebra. Moreover, Ahn and Kim [8] introduced the notion of QS-algebra which is a paper subclass of Q-algebra. In 2013 [2], introduced a new notion called KUS-algebra, which is dual for QS-algebras and investigated several basic properties which are related to KUS-ideal. In this paper, we introduce the notion of fuzzy KUS-ideals in KUS-algebras and then we investigate several basic properties which are related to fuzzy KUS-ideals. We describe how to deal with the homomorphism of image and inverse image of fuzzy KUS-ideals of KUS-algebras.

2. Preliminaries

Definition 2.1([2]). Let (X; * ,0) be an algebra with a single binary operation (*). X is called a KUS-algebra if it satisfies the following identities: for any x, y, z ∈ X, (kus_1): (z * y) * (z * x) = y * x, (kus_2): 0 * x = x, (kus_3): x * x = 0, (kus_4): x * (y * z) = y * (x * z).

In X we can define a binary relation (≤) by: x ≤ y if and only if y * x = 0.

Lemma 2.2 ([2]). In any KUS-algebra (X;*,0), the following properties hold: for all x, y, z ∈ X;

a) x * y = 0 and y * x = 0 imply x = y,
b) y * [(y * z) * z] = 0,
c) (0 * x) * (y * x) = y * 0,
d) (x * y) * 0 = y * x.

Theorem 2.3([2]). Any KUS-algebra is equivalent to the dual QS-algebra.

Definition 2.4([2]). Let X be a KUS-algebra and let S be a nonempty subset of X. S is called a KUS-sub-algebra of X if x * y ∈ S whenever x ∈ S and y ∈ S.
Definition 2.5 ([2]). A nonempty subset I of a KUS-algebra X is called a KUS-ideal of X if it satisfies: for x, y, z \in X,

(Ikus_1) \ (0 \in I), 
(Ikus_2) \ ((z * y) \in I \text{ and } (y * x) \in I \text{ imply } (z * x) \in I). 

Example 2.6. Let X = \{0, a, b, c\} in which (* ) is defined by the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
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</table>

Then (X; *, 0) is a KUS-algebra. It is easy to show that I_1 = \{0, a\}, I_2 = \{0, b\}, I_3 = \{0, c\}, and I_4 = \{0, a, b, c\} are KUS-ideals of X.

Proposition 2.7([2]). Every KUS-ideal of KUS-algebra X is a KUS-sub-algebra.

3. Fuzzy KUS-ideals and Homomorphism of KUS-algebras

In this section, we will discuss a new notion called fuzzy KUS-ideals of KUS-algebras and study several basic properties which are related to fuzzy KUS-ideals.

Definition 3.1([6]). Let (X; *, 0) be a nonempty set, a fuzzy subset \( \mu \) in X is a function \( \mu: X \rightarrow [0,1] \).

Definition 3.2. Let (X; *, 0) be a KUS-algebra, a fuzzy subset \( \mu \) in X is called a fuzzy KUS-sub-algebra of X if for all x, y \in X, \( \mu(x * y) \geq \min \{\mu(x), \mu(y)\} \).

Definition 3.3. Let (X; *, 0) be a KUS-algebra, a fuzzy subset \( \mu \) in X is called a fuzzy KUS-ideal of X if it satisfies the following conditions: for all x, y, z \in X,

(Fkus_1) \( \mu(0) \geq \mu(x) \), 
(Fkus_2) \( \mu(z * x) \geq \min \{\mu(z * y), \mu(y * x)\} \).

Example 3.4.

1) Let X = \{0, 1, 2, 3\} in which (*) is defined by the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
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<th>3</th>
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</tbody>
</table>

Then (X; *, 0) is a KUS-algebra. Define a fuzzy subset \( \mu: X \rightarrow [0,1] \) by

\( \mu(x) = \begin{cases} 0.7 & \text{if } x \in \{0,1\} \\ 0.3 & \text{otherwise} \end{cases} \)

I_1 = \{0, 1\} is a KUS-ideal of X. Routine calculation gives that \( \mu \) is a fuzzy KUS-ideal of KUS-algebras X.

2) Consider X = \{0, a, b, c, d\} with (*) defined by the table:

<table>
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<th>b</th>
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</table>

Then (X; *, 0) is a KUS-algebra. Define a fuzzy subset \( \mu: X \rightarrow [0,1] \) such that

\( \mu(0) = t_1, \mu(a) = \mu(b) = \mu(c) = \mu(d) = t_2 \), \( ,\text{where } t_1, t_2 \in [0, 1] \) and \( t_1 > t_2 \).

Routine calculation gives that \( \mu \) is a fuzzy KUS-ideal of KUS-algebra X.

Definition 3.5 ([6]). Let X be a nonempty set and \( \mu \) be a fuzzy subset in X, \( \mu \) is a fuzzy KUS-sub-algebra of X.

Definition 3.6. Let \( \mu \) be a fuzzy KUS-ideal in KUS-algebra X, \( \mu \) is a fuzzy KUS-ideal of X if and only if, for every \( t \in [0,1] \), \( \mu_t \) is either empty or a KUS-ideal of X.

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Proof: Assume that $\mu$ is a fuzzy KUS-ideal of $X$, by (Fkus$_1$), we have $\mu(0) \geq \mu(x)$ for all $x \in X$ therefore $\mu(0) \geq \mu(x) \geq t$ for $x \in \mu_1$ and so $0 \in \mu_1$. Let $x, y, z \in X$ be such that 
\[(z \ast y) \in \mu_1 \text{ and } (y \ast x) \in \mu_1, \text{ then} \]
\[\mu(z \ast y) \geq t \text{ and } \mu(y \ast x) \geq t, \text{ since } \mu \text{ is a} \]
fuzzy KUS-ideal, it follows that 
\[\mu(z \ast x) \geq \min \{\mu(z \ast y), \mu(y \ast x)\} \geq t \text{ and we have that } x \ast z \in \mu_1. \text{ Hence } \mu_1 \text{ is a} \]
KUS-ideal of $X$.

Conversely, we only need to show that (Fkus$_1$) and (Fkus$_2$) are true. If (Fkus$_1$) is false, then there exist $x \in X$ such that 
\[\mu(0) < \mu(x'). \]
If we take 
\[t' = (\mu(x') + \mu(0))/2, \text{ then } \mu(0) < t' \text{ and} \]
\[0 \leq t' < \mu(x') \leq 1, \text{ then } x' \in \mu \text{ and } \mu \neq \emptyset. \]
As $\mu_1$ is a KUS-ideal of $X$, we have 
\[0 \in \mu_1 \text{ and so } \mu(0) \geq t'. \text{ This is a} \]
contradiction. Now, assume (Fkus$_2$) is not true, then there exist $x' \in \mu_1$ such that 
\[\mu(z' \ast x') < \min \{\mu(z' \ast y'), \mu(y' \ast x')\}. \]

Putting 
\[t'=\mu(z' \ast x') + \min \{\mu(z' \ast y'), \mu(y' \ast x')\}/2, \text{ then } \mu(x' \ast x') < t' \text{ and} \]
\[0 \leq t' < \min \{\mu(z' \ast x'), \mu(y' \ast x')\} \leq 1, \text{ hence} \]
\[\mu(z' \ast y')) > t' \text{ and } \mu(y' \ast x') > t', \text{ which imply that} \]
\[(z' \ast y') \in \mu_1 \text{ and } (y' \ast x') \in \mu_1. \]

Since $\mu_1$ is a KUS-ideal, it follows that 
\[(x' \ast z') \in \mu_1 \text{ and that } \mu(x' \ast z') \geq t', \text{ this} \]
is also a contradiction. Hence $\mu$ is a fuzzy KUS-ideal of $X$. □

Corollary 3.7. Let $\mu$ be a fuzzy subset in KUS-algebra $X$. If $\mu$ is a fuzzy KUS-ideal, then for every $t \in \operatorname{Im}(\mu)$, $\mu_t$ is a KUS-ideal of $X$ when $\mu_t \neq \emptyset$.

Theorem 3.8. Let $\mu$ be a fuzzy subset in KUS-algebra $X$. If $\mu$ is a fuzzy KUS-sub-algebra of $X$ if and only if, for every $t \in [0,1]$, $\mu_t$ is either empty or a KUS-sub-algebra of $X$.

Proof: Assume that $\mu$ is a fuzzy KUS-sub-algebra of $X$, let $x, y \in X$ be such that $x \in \mu_1$ and $y \in \mu_1$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Since $\mu$ is a fuzzy KUS-sub-algebra, it follows that 
\[\mu(x \ast y) \geq \min \{\mu(x), \mu(y)\} \geq t \text{ and that} \]
\[(x \ast y) \in \mu_1. \]

Hence $\mu_1$ is a KUS-sub-algebra of $X$.

Conversely, assume 
\[\mu(x \ast y) \geq \min \{\mu(x), \mu(y)\} \text{ is not true}, \text{ then there exist } x' \text{ and } y' \in X \text{ such that} \]
\[\mu(x' \ast y') < \min \{\mu(x'), \mu(y')\}. \]

Putting 
\[t'=(\mu(x' \ast y') + \min \{\mu(x'), \mu(y')\}/2, \text{ then } \mu(x') < t' \text{ and} \]
\[0 \leq t' < \min \{\mu(x'), \mu(y')\} \leq 1, \text{ hence} \]
\[\mu(x') > t' \text{ and } \mu(y') > t', \text{ which imply that} \]
\[x' \in \mu_1 \text{ and } y' \in \mu_1, \text{ since } \mu_1 \text{ is a} \]
KUS-sub-algebra, it follows that 
\[x' \ast y' \in \mu_1 \text{ and } \mu(x' \ast y') \geq t', \text{ this is also a contradiction. Hence } \mu \text{ is a fuzzy KUS-sub-algebra of } X. \triangleq

Proposition 3.9. Every fuzzy KUS-ideal of KUS-algebra $X$ is a fuzzy KUS-sub-algebra of $X$.

Proof: Since $\mu$ is fuzzy KUS-ideal of a KUS-algebra $X$, then by theorem (3.6), for every $t \in [0,1]$, $\mu_t$ is either empty or a KUS-ideal of $X$. By proposition(2.7), for every $t \in [0,1]$, $\mu_t$ is either empty or a KUS-sub-algebra of $X$. Hence $\mu$ is a fuzzy KUS-sub-algebra of KUS-algebra $X$ by theorem (3.8). □

Definition 3.10 ([1]). Let $(X; \ast, 0)$ and $(Y; \ast', 0')$ be nonempty sets. The mapping $f : (X; \ast, 0) \rightarrow (Y; \ast', 0')$ is called a homomorphism if it satisfies: 
\[f(x \ast y) = f(x) \ast' f(y), \text{ for all } x, y \in X. \]

The set \( \{x \in X \mid f(x) = 0'\} \) is called the Kernel of $f$ denoted by Ker $f$. □
Definition 3.11 ([11]). Let \( f : (X; *, 0) \rightarrow (Y; *, 0') \) be a mapping nonempty sets \( X \) and \( Y \) respectively. If \( \mu \) is a fuzzy subset of \( X \), then the fuzzy subset \( \beta \) of \( Y \) defined by:

\[
f(\mu(f^{-1}(y))) = \begin{cases} \sup \{\mu(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
\]

is said to be the image of \( \mu \) under \( f \).

Similarly if \( \beta \) is a fuzzy subset of \( Y \), then the fuzzy subset \( \mu = (\beta \circ f) \) in \( X \) i.e the fuzzy subset defined by \( \mu(x) = \beta(f(x)) \) for all \( x \in X \) is called the pre-image of \( \beta \) under \( f \).

Theorem 3.12. An into homomorphism pre-image of a fuzzy KUS-ideal is also a fuzzy KUS-ideal.

Proof: Let \( f : (X; *, 0) \rightarrow (Y; *, 0') \) be an into homomorphism of KUS-algebras, \( \beta \) a fuzzy KUS-ideal of \( Y \) and \( \mu \) the pre-image of \( \beta \) under \( f \), then \( \beta(f(\mu)) = \mu(x) \) for all \( x \in X \).

Since \( f(x) \in Y \) and \( \beta \) is a fuzzy KUS-ideal of \( Y \), it follows that \( \beta(0') = \beta(\mu(f(0'))) = \mu(0) \), for every \( x \in X \), where \( 0' \) is the zero element of \( Y \).

But \( \mu(0) \geq \mu(0) \) and so \( \mu(0) \geq \mu(x) \) for \( x \in X \).

Now let \( x, y, z \in X \), then we get

\[
\beta(z * x) = \beta(f(z * x)) = \beta(f(z) * f(x)) \\
\geq \min \{\beta(f(z)), \beta(f(x))\} \\
= \min \{\beta(f(z * y)), \beta(f(y * x))\}
\]

i.e., \( \mu(z * x) \geq \min \{\mu(z * y), \mu(y * x)\} \) for all \( x, y, z \in X \).

Definition 3.13 ([11]). A fuzzy subset \( \mu \) of a set \( X \) has sup property if for any subset \( T \) of \( X \), there exist \( t_0 \in T \) such that \( \mu(t_0) = \sup \{\mu(t) : t \in T\} \).

Theorem 3.14. Let \( f : (X; *, 0) \rightarrow (Y; *, 0') \) be a homomorphism between KUS-algebras \( X \) and \( Y \) respectively. For every fuzzy KUS-ideal \( \mu \) in \( X \) with sup property, \( f(\mu) \) is a fuzzy KUS-ideal of \( Y \).

Proof: By definition \( \beta(y') = f(\mu(y')) := \sup \{\mu(x) \mid x \in f^{-1}(y')\} \), for all \( y' \in Y \) (sup \( \emptyset = 0 \)).

We have to prove that \( \beta(z * x') \geq \min \{\beta(z * y'), \beta(y' * x')\} \), for all \( x', y', z' \in Y \).

(I) Let \( f : (X; *, 0) \rightarrow (Y; *, 0') \) be a homomorphism of KUS-algebras, \( \mu \) a fuzzy KUS-ideal of \( X \) with sup property and \( \beta \) the image of \( \mu \) under \( f \). Since \( \mu \) is a fuzzy KUS-ideal of \( X \), we have \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Note that \( 0 \in f^{-1}(0') \), where \( 0' \) is the zero element of \( Y \). Thus \( \beta(0') = \sup \{\mu(t) : t \in f^{-1}(0')\} \geq \mu(0) \geq \mu(x) \) for all \( x \in X \), which implies that \( \beta(0') \geq \sup \{\mu(t) : t \in f^{-1}(x')\} \).

For any \( x', y', z' \in Y \), let \( x_0 \in f^{-1}(x') \). Then \( y_0 \in f^{-1}(y') \), \( z_0 \in f^{-1}(z') \) be such that:

\[
\mu(z_0 * y_0) = \beta(f(z_0 * y_0)) = \beta(f(z') * y') = \sup \{\mu(t) : t \in f^{-1}(z') * y'\} = \sup \{\mu(t) : t \in f^{-1}(z') * y'\} = \beta(z' * y') \geq \beta(z' * y') \geq \mu(z_0 * y_0)
\]

Hence \( \beta \) is a fuzzy KUS-ideal of \( Y \).
(II) If \( f \) is not onto: For every \( x^{\dagger} \in Y \), we define \( X_{x^{\dagger}} := f^{-1}(x^{\dagger}) \). Since \( f \) is a homomorphism, we get
\[
X_{x^{\dagger}} \ast X_{y^{\dagger}} \subseteq X_{x^{\dagger} \ast y^{\dagger}}, \text{ and } X_{y^{\dagger}} \ast X_{x^{\dagger}} \subseteq X_{y^{\dagger} \ast x^{\dagger}},
\]
for all \( x^{\dagger}, y^{\dagger}, z^{\dagger} \in Y \)  \\
Let \( x^{\dagger}, y^{\dagger}, z^{\dagger} \in Y \) be arbitrarily given. If
\[
(z^{\dagger} \ast y^{\dagger}) \not\in \text{Im}(f) = f(X), \text{ then by definition } \beta(z^{\dagger} \ast y^{\dagger}) = 0 . \text{ But if }
(z^{\dagger} \ast y^{\dagger}) \not\in f(X), \text{ i.e., } X_{z^{\dagger} \ast y^{\dagger}} = \phi \text{, then by (*) at least one of } z^{\dagger}, y^{\dagger}, x^{\dagger} \not\in f(X) \text{ and hence } \beta(x^{\dagger} \ast x^{\dagger}) \geq 0 = \min \{ \beta(x^{\dagger} \ast y^{\dagger}), \beta(y^{\dagger} \ast x^{\dagger}) \}. \triangle

References


