Abstract
This paper analyses the local behavior of the cubic function approximation of the form \( P(z) f(z) + Q(z) f(z) + R(z) = O(z^{p+q+r+2}) \), where \( P(z), Q(z), R(z) \) are algebraic polynomials of degree \( p, q, r \) respectively, to a function which has a given power series expansion about the origin. It is shown that the cubic Hermite-Padé form always defines a cubic function and that this function is analytic in a neighbourhood of the origin.

Keywords: Cubic function approximation, Hermite-Padé approximation, algebraic polynomials

1. Introduction

The Padé approximation theory has been widely used in problems of theoretical physics [1, 3], numerical analysis [6] [7], and electrical engineering, especially in modal analysis model [2], order reduction of multivariable systems [4, 8].

Let \( f(z) \) be a function, analytic in some neighbourhood of the origin, whose series expansion about the origin is known. In this paper we wish to consider the properties of the cubic Hermite-Padé approximations to \( f(z) \) generated by finding polynomials \( P(z), T(z), Q(z) \) and \( R(z) \) such that

\[
P(z) f^3(z) + T(z) f^2(z) + Q(z) f(z) + R(z) = O(z^{p+q+r+3}),
\]

with \( P(z), T(z), Q(z), R(z) \) being algebraic polynomials of degree \( p, t, q, r \) respectively. But as is well known, if we set

\[
z = y - \frac{a}{3},
\]

then any cubic equation

\[
z^3 + az^2 + bz + c = 0,
\]

can be transformed into the following form

\[
y^3 + (b - \frac{a^2}{3})y + \left(\frac{2}{27}a^3 - \frac{1}{3}ab + c\right) = 0.
\]

So without loss of generality, in this paper we only consider approximations to \( f(z) \) generated by finding polynomials \( P(z), Q(z) \) and \( R(z) \) so that

\[
P(z) f^3(z) + Q(z) f(z) + R(z) = O(z^{p+q+r+2}). \tag{1}
\]

Note that such polynomials \( P, Q, R \) not all zero, must exist since (1) represents a homogenous system of \( p+q+r+2 \) linear equations in the \( p+q+r+3 \) unknown coefficients of the \( P(z), Q(z), R(z) \). Then set

\[
P(z) u^3(z) + Q(z) u(z) + R(z) = 0
\]

and attempt to solve this equation for \( u(z) \) in such a way that \( u(z) \) approximates \( f(z) \).

In the well-known case of Padé approximation [1], the same procedure is followed by

\[
P(z) f(z) + Q(z) = O(z^{p+q+1})
\]

which gives

\[
u(z) = -\frac{Q(z)}{P(z)}.
\]

If \( P(0) \neq 0 \) (not a serious restriction), it then follows that

\[
u(z) = f(z) + O(z^{p+q+1}).
\]

In the case of quadratic Hermite-Padé approximation [5], the procedure is followed by

\[
P(z) u^2(z) + Q(z) u(z) + R(z) = O(z^{p+q+r+2})
\]

which gives

\[
u(z) = (-Q(z) \pm \sqrt{B(z)}) / (2P(z)),
\]

where

\[
B(z) = Q^2(z) - 4P(z)R(z).
\]

If \( B(0) \neq 0 \), it then follows that

\[
u(z) = f(z) + O(z^{p+q+r+2}).
\]

If \( B(0) = 0 \), we set

\[
B(z) = e^2 g(z), g(0) \neq 0
\]

(since Ref. [5] has proved that \( B(z) \) never has a root of odd multiplicity at the origin). It then follows that

\[
u(z) = f(z) + O(z^{p+q+r+2-s}),
\]

where \( 2s < p+q+r+1 \).

However, in the cubic case it is not obvious that

\[
P(z) u^3(z) + Q(z) u(z) + R(z) = 0
\]

yields even an analytic approximation to \( f(z) \), still less that it defines a function \( u(z) \) such that

\[
u(z) = f(z) + O(z^{p+q+r+2}).
\]

The purpose of this paper is to show that an analogue of the Padé and quadratic Hermite-Padé results is in fact true.

2. Notation

It is assumed that

\[
P(z) f^3(z) + Q(z) f(z) + R(z) = O(z^{N+2}),
\]

where \( N \geq p+q+r \) and that

\[
|P(0)| + |Q(0)| + |R(0)| \neq 0.
\]
Note that if $z^{\ell}$ is the maximal common factor of $P(z), Q(z), R(z)$, then
\[ \frac{P(z)}{z^{\ell}} f^{\ell}(z) + \frac{Q(z)}{z^{\ell}} f(z) + \frac{R(z)}{z^{\ell}} = O(z^{n-2}) \]
so that this second assumption is not a serious restriction.

The following notation will be used:
(i) An approximation derived from
\[ P(z) f(z) + Q(z) f(z) + R(z) = O(z^{n-2}) \]
will be referred to as a $(p,q,r)$ cubic approximation to $f(z)$.
(ii) Let
\[ D(z) = \frac{1}{4} P(z) R^2(z) + \frac{1}{27} Q^3(z). \]
(iii) By $\sqrt[3]{D(z)}, \sqrt[3]{E(z)}$ we mean the principal square root of $D(z), E(z)$ respectively.

3. The Principal Results

The problem divides itself into two cases, the case $D(0)=0$ and the case $D(0) \neq 0$.

3.1 The Case $D(0) \neq 0$

Theorem 1. If $D(0) \neq 0$, then there exists a unique function $u(z)$, analytic in a neighbourhood of the origin, satisfying
\[ P(z) f(z) + Q(z) f(z) + R(z) = 0 \]
and $u(0)=f(0)$.

Proof. (i) Suppose $P(0)Q(0) \neq 0$. The three possible expressions for $u(z)$ in a neighbourhood of the origin are given by
\[ u(z) = \omega_k \left[ \left( \frac{R(z)}{2P(z)} + \sqrt{\frac{D(z)}{P^3(z)}} \right) - \omega_k \left( \frac{R(z)}{2P(z)} + \sqrt{\frac{D(z)}{P^3(z)}} \right), \right. \]
where
\[ \omega_k = \frac{-1 + \sqrt{3i}}{2}, k = 0,1,2; \]
\[ \omega_k = \frac{-1 - \sqrt{3i}}{2}. \]

Since $P(0)Q(0)D(0) \neq 0$, these three functions are all analytic in a neighbourhood of the origin. Exactly one of them satisfies $u(0)=f(0)$, because
\[ P(0) f^{\ell}(0) + Q(0) f(0) + R(0) = 0 \]
so $f(0) = \omega_k \left[ \left( \frac{R(0)}{2P(0)} + \sqrt{\frac{D(0)}{P^3(0)}} \right) - \omega_k \left( \frac{R(0)}{2P(0)} + \sqrt{\frac{D(0)}{P^3(0)}} \right) \right], k = 0,1,2.$

(ii) Suppose $Q(0)=0$. Then $P(0) \neq 0$ (since $D(0) \neq 0$). The three possible expressions for $u(z)$ in a neighbourhood of the origin are given by
\[ u(z) = \omega_k \left[ \left( \frac{R(z)}{2P(z)} + \sqrt{\frac{D(z)}{P^3(z)}} \right) - \omega_k \left( \frac{R(z)}{2P(z)} + \sqrt{\frac{D(z)}{P^3(z)}} \right) \right], k = 0,1,2. \]

Since $P(0)D(0) \neq 0$, these three functions are all analytic in a neighbourhood of the origin. Also exactly one of them satisfies $u(0)=f(0)$, because
\[ P(0) f^{\ell}(0) + Q(0) f(0) + R(0) = 0 \]
so $f(0) = \omega_k \left[ \left( \frac{R(0)}{2P(0)} + \sqrt{\frac{D(0)}{P^3(0)}} \right) - \omega_k \left( \frac{R(0)}{2P(0)} + \sqrt{\frac{D(0)}{P^3(0)}} \right) \right], k = 0,1,2.$

(iii) Suppose $Q(0) \neq 0$, $P(0)=0$.

Near the origin the three possible expressions
\[ u_k(z) = \frac{-1 + \sqrt{3i}}{2}, k = 0,1,2; \]
\[ u_k(z) = \frac{-1 - \sqrt{3i}}{2}. \]

The right-hand sides of $u_k(z)$ $(k=0,1,2)$ are unbounded as $z \to 0$, so we can exclude these possibilities. Since $P(0)=0$, close to the origin we can apply the binomial theorem to get from $u_k(z)$ the convergent power series (analytic in a neighbourhood of the origin) expression for $u(z)$.

It follows that
\[ u(z) = \left\{ \begin{array}{ll}
\frac{R(z)}{2P(z)} + \sqrt{\frac{D(z)}{P^3(z)}} - \frac{3R(z)}{Q^3(z)}, & z \neq 0 \\
-\frac{3R(z)}{Q^3(z)}, & z = 0
\end{array} \right. \]
is the only function, analytic in a neighbourhood of the origin, satisfying
\[ P(z) u^{\ell}(z) + Q(z) u(z) + R(z) = 0 \]
with $u(0)=f(0)$.

Theorem 2. If $Q(0)D(0) \neq 0$, then there exists a unique function $u(z)$, analytic in a neighbourhood of the origin, satisfying
\[ P(z)u'(z) + Q(z)u(z) + R(z) = 0 \]
such that
\[ u(z) = f(z) + O(z^{N+2}). \]

**Proof.** Note that
\[ \frac{d^j}{dz^j} \left[ P(z)u'(z) + Q(z)u(z) + R(z) \right] = 0, \]
\[ j \in \{0, 1, \ldots, N+1\}. \]

For \( j=1 \)
\[ \left[ \frac{\partial}{\partial u} \left( P(z)u'(z) + Q(z)u(z) + R(z) \right) u'(z) \right] = 0, \]
\[ \left[ \frac{\partial}{\partial f} \left( P(z)f'(z) + Q(z)f(z) + R(z) \right) f'(z) \right] = 0. \]

Differentiating again \((j=2)\) gives
\[ \left[ \frac{\partial}{\partial u} \left( P(z)u'(z) + Q(z)u(z) + R(z) \right) u''(z) \right] = 0, \]
\[ \left[ \frac{\partial}{\partial f} \left( P(z)f'(z) + Q(z)f(z) + R(z) \right) f''(z) \right] = 0. \]

In general, more compact form we have
\[ \left[ \frac{\partial}{\partial u} \left( P(z)u'(z) + Q(z)u(z) + R(z) \right) u^{(j)}(z) + u_j(z) \right] = 0, \]
\[ j \in \{1, 2, \ldots, N+1\}. \]

where
\[ u_1(z) = P'(z)u'(z) + Q'(z)u(z) + R'(z), \]
\[ u_{j+1}(z) = \frac{d}{dz} u_j(z) + \frac{d}{dz} \left[ \frac{\partial}{\partial u} \left( P(z)u'(z) + Q(z)u(z) + R(z) \right) \right] u_j^{(j)}(z); \]
\[ v_1(z) = P'(z)f'(z) + Q'(z)f(z) + R'(z), \]
\[ v_{j+1}(z) = \frac{d}{dz} v_j(z) + \frac{d}{dz} \left[ \frac{\partial}{\partial f} \left( P(z)f'(z) + Q(z)f(z) + R(z) \right) \right] f_j^{(j)}(z). \]

Now taking the unique \( u(z) \) from Theorem 1, it is easily proved that
\[ \frac{d}{du} \left[ P(z)u'(z) + Q(z)u(z) + R(z) \right] \neq 0, \]
since
\[ D(0) = \left[ \frac{1}{4} P(z)R^2(z) + \frac{1}{27} Q'(z) \right] \neq 0, \]
and
\[ \left[ P(z)u'(z) + Q(z)u(z) + R(z) \right] \neq 0. \]

Therefore it is seen that Eq. (4) with \( j=1 \) gives \( u'(0) = f'(0) \), which with \( j=2 \) gives \( u''(0) = f''(0) \).

It follows that
\[ u^{(j)}(0) = f^{(j)}(0), \quad j \in \{1, 2, \ldots, N+1\}, \]
i.e.
\[ u(z) = f(z) + O(z^{N+2}). \]

3.2 The Case \( D(0)=0 \).

We now investigate the case \( D(0)=0 \). This implies that \( P(0) \neq 0 \) (since if
\[ D(0) = \frac{1}{4} P(0)R^2(0) + \frac{1}{27} Q'(0) = 0 \]
and \( P(0)=0 \), then \( Q(0)=0 \), which with \( P(0)f'(0) + Q(0)f(0) + R(0) = 0 \)
gives \( R(0)=0 \); this contradicts the assumption that \( |P(0)| + |Q(0)| + |R(0)| \neq 0 \).

First, it is necessary to treat two special cases:
(i) Suppose \( R(z) \equiv 0 \).

Then
\[ P(z)f'(z) + Q(z)f(z) = O(z^{N+2}) \]
\[ \Rightarrow (P(z)f^2(z) + Q(z)) f(z) = O(z^{N+2}) \]
so that
\[ f^2(z) = -Q(z)/P(z) + O(z^S), \]
and
\[ f(z) = O(z^T), \quad \text{where } S + T = N + 2. \]

Choosing
\[ u(z) = \begin{cases} \sqrt{-Q(z)/P(z)}, & \text{if } S > 2T \\ 0, & \text{otherwise} \end{cases} \]
gives \( u(z) \) such that
\[ P(z)u'(z) + Q(z)u(z) + R(z) = 0 \]
and
\[ u(z) = f(z) + O(z^{\text{max}(S,2T)}). \]

Clearly \( \max\{S, 2T\} \geq (N + 2)/3 \).

(ii) Suppose \( Q(z) \equiv 0 \).

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Then
$$P(z) f^3(z) + R(z) = O(z^{N+2})$$
$$\Rightarrow -R(z) / P(z) = f^3(z) + O(z^K), \quad K \geq (N+2) / 3.$$  
so that
$$u(z) = -\sqrt{R(z) / P(z)} = f(z) + O(z^K), \quad K \geq (N+2) / 3.$$  
and
$$P(z) u'(z) + Q(z) u(z) + R(z) = 0.$$

**Theorem 3.** If \( D(z) \neq 0 \) and \( R(z) \neq 0 \), then \( D(z) \) never has a root of multiplicity greater than \( p+q+2r \) at the origin.

**Proof.** Let \( M = p+q+2r \) and suppose \( D(z) = z^{M+1} D_1(z) \), where \( D_1(z) \) is a polynomial of degree \( s \). Since \( P(z) \neq 0, R(z) \neq 0 \), then
$$Q'(z) = 27 z^{M+1} D_1(z) + B_1(z), \quad (5)$$
where \( B_1(z) \) is a nonzero polynomial of degree \( t \). We must have \( M + l + s = 3q \) (since \( p + 2r \leq M < M+1 \)) so that
$$B_1(z) = \frac{27}{4} P(z) R^2(z).$$

Also \( t + q = p + q + 2r < M + 1 = 3q - s \Rightarrow t + s < 2q \). Differentiating (5) gives
$$3Q'(z) Q''(z) = 27z^{M} \left( (M+1) D_1(z) + zD'_1(z) \right) + B'_1(z)$$
$$\Rightarrow 3zQ'(z) Q''(z) = 27z^{M+1} \left( (M+1) D_1(z) + zD'_1(z) \right) + zB'_1(z)$$
$$= 27z^{M+1} D_j(z) + B_j(z), \quad (6)$$
where
$$D_j(z) = (M+1) D_1(z) + zD'_1(z) \quad \text{(degree } s)$$
$$B_j(z) = zB'_1(z) \quad \text{(degree } t)$$

From (5) and (6) (eliminating the term in \( z^{M+1} \)) we have
$$Q'(z) \left[ D_j(z) Q(z) - 3D_1(z) zQ'(z) \right] = B_1(z) D_j(z) - B_j(z) D_1(z). \quad (7)$$

The left-hand side of (7) either has degree not less than \( 2q \) or is identically zero, while the right-hand side has degree not greater than \( t + s < 2q \). It follows that
$$D_j(z) Q(z) - 3D_1(z) zQ'(z) = 0 = B_1(z) D_j(z) - B_j(z) D_1(z).$$

Hence
$$\frac{Q'(z)}{Q(z)} = \frac{D_j(z)}{3zD_1(z)} = \frac{B_j(z)}{3zB_1(z)} = \frac{B'_1(z)}{3B_1(z)}.$$  
And integrating gives
$$Q(z) = c \sqrt{B_1(z)}, \quad c \in \mathbb{R}.$$  
But
$$\text{deg } \sqrt{B_1(z)} = t / 3 < q$$
so the result is proved by contradiction.