"X-Vaults": a Software for the Analysis of the Stability of Masonry Cross-Vaults

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Abstract

Since 2003, when a stunned Italy witnessed the collapse of the school in San Giuliano di Puglia (CB) due to a major seismic event, Italian technical regulations for constructions have been subjected to continual changes and additions, until the adoption of DM 14-01-2008. It highlights the Italian trend to move towards Eurocodes. The new regulations, for how they were conceived, require professional architects and engineers structural calculations, which can not be carried out by hand, without the aid of a special structural software in Civil Engineering. In this paper the authors would like to present a software for the analysis of the stability of masonry cross-vaults, deriving from their own scientific research and which has been put at everyone's disposal by a software-house which has shown immediately interest in the proposed topic.

Keywords: *Structural Analysis Software, Numerical Algorithm, Generalized Inverse, Cross-Vaults, Masonry, Non-Linear Behaviour.*

1. Introduction

The topic of the analysis of masonry structures is a very delicate issue of Civil Engineering, because "masonry material" which makes up most of the Italian buildings is, in some ways, still unknown both from a behavioural point of view and from the point of view of its mechanical properties, because of the number and different types of its components (bricks, stones, mortars). In one sense it is not also so correct to refer to masonry using the term "material".

While the behaviour of other materials, such as steal and metal alloys, are perfectly described by the laws of Structural Mechanics, the behaviour of masonry is still studied by Italian and foreign researchers and can be considered as an open research topic.

In fact, the mechanical behaviour of masonry is strongly influenced by the number and kind of its components (inert materials), from the arrangement, the size and the orientation of the mortar joints (binder) that confine and bind the inert blocks together, from the age of the building and its life history (changes that may have altered the static behaviour with the passing of the time).

Moreover, a new masonry construction is further different from another one existing for centuries, both for the arrangement of blocks in the masonry apparatus and for the role itself of the mortar: in the first case, the more or less equal blocks are arranged in a tidy manner and the mortar is generally in good conditions, presenting good elastic and cohesive properties; in the second case, the blocks, with different shapes, sizes and materials, are often chaotically arranged and the role of the mortar is rather uncertain, modified by time and mainly by hydration that has made it powdery and inconsistent.

According to the above, the authors, who have devoted to the study of statics and stability of monumental and masonry constructions for years, prefer to deal with the structural problem of ancient masonry pivoting on the following hypotheses:

- masonry blocks can be considered rigid, that is they do not suffer elastic deformations when subjected to external actions (such as loads and constraint displacements);
- mortar, aged because of the passing of time, or not present since the planning of the construction, does not play a key role in the behaviour of the structure, so that you can consider the structure as dry assembled.

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This article will deal with the issue of special masonry structures which have been used extensively in the roofing of medieval buildings, such as in the ceilings of the aisles in Romanesque cathedrals and Gothic abbeys, historical, architectonical and cultural heritage all over the world. This kind of structures is called *cross-vault* or *groin vault* too, because it is the geometric result of the crossing of two barrel vaults.

2. Cross-Vaults: Geometrical Construction and Structural Behaviour

The geometric construction of a cross-vault (Fig. 1) is relatively simple. Let us consider the shape of an arch (generatrix). When it moves parallely to itself, along a line (directrix, generally a straight line orthogonal to the plane of the arch), you obtain a translational surface, that is a barrel vault. Afterwards the barrel vault is cut by two vertical planes, so as to subdivide the vault itself into four parts, equal two by two: the cells kind 1 and the cells kind 2 [1].

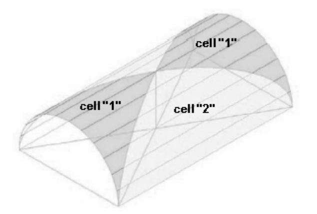

Fig. 1 Decomposition of a barrel vault to obtain a cross one.

At last the cross-vault is obtained merging a certain number of cells kind 1 (in the following text: cells), at least four of them placed at right angles to obtain a squarish unit.

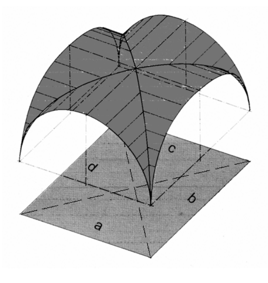

Fig. 2 Cross-vault obtained merging four cells.

The above mentioned geometric construction is the key secret to understand as better as possible the structural behaviour of a cross-vault.
According to the instructions suggested in the nineteenth century by Mèry [2], the authors conclude that *the behaviour of a cross-vault depends on the structural behaviour of all the portions in which it is possible to subdivide it, following a backward geometric decomposition, beginning from the whole vault to obtain the generating arch.* Thus, observing Fig. 2, let us decompose the vault in its four constituent portions (cells), which touch each other along two diagonal curve surfaces, named *diagonal arches* (or even *ribs* in Gothic architecture). Finally, remembering that each cell is a portion of a barrel vault, the process is concluded with the further decomposition of the cell in a series of *elemental arches*, whose depth is arbitrary chosen (Fig. 3).

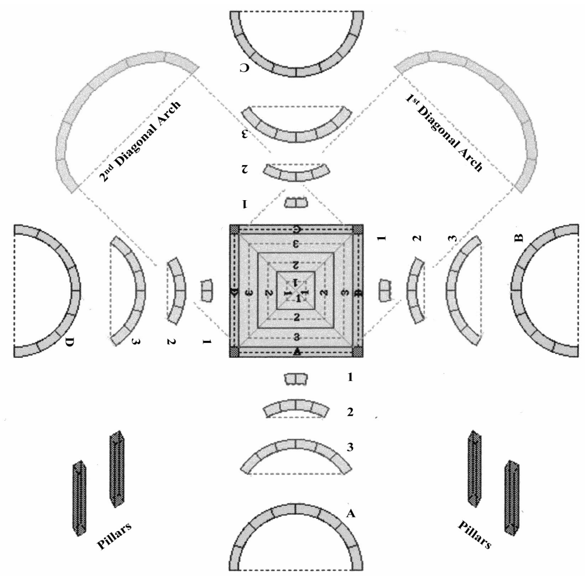

Fig. 3 Back decomposition of the cross-vault to create elemental arches.

Following this logical thread, a cross-vault is based on the structural principle of an arch: all the elemental arches forming the cells support the upper loads in the reason of their depth and transfer them to the diagonal arches, which, in turn, lead them to their springs and consequently to the piers placed at the four corners on which the vault is set.
Finally, to satisfy its basic conception of existence, that is its stability, a cross-vault is always edged along the perimeter by additional transverse arches or walls. These last boundary structures are necessary and provide assistance to the maintenance of the equilibrium of the vault; in fact, they are called to support the out-of-plane thrusts coming from the elemental arches.

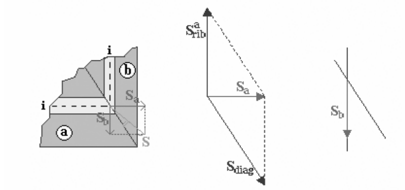

Fig. 4 Vectorial decomposition.

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Observing Fig. 4, let us consider each pair of elemental arches of the same name *i* (that is identified by the same number, 1 with 1, 2 with 2, 3 with 3, etc.) belonging to two different cells (*a* and *b*) crossing on the same diagonal arch. According to the authors' point of view, if S_a is the thrust at the spring provided by the *i-th* elemental arch belonging to the cell "a" and S_b the thrust at the spring provided by the *i-th* elemental arch belonging to the cell "b", by the vectorial decomposition above shown it is possible to quantify the rate S_{diag} of such forces acting on the diagonal arch and the rate S_{rib} acting on the transverse arch.
Thus all the loads acting on each sub-structure obtained from the decomposition of the vault is clearly quantified: the elemental arches belonging to the cells are subjected to vertical loads due to their depth and transmit, through their spring interface, vertical and horizontal thrusts both to the diagonal arches and to the transverse arches.
To establish if a cross-vault, subjected to a certain load condition, is stable, we propose a numerical algorithm to process all these substructures. The numerical procedure states "stable" or "unstable" each substructure and, only if they all have been stated stable, it is possible to affirm the whole cross-vault is stable.

3. The Structural Model of the Generating Arch

The structural model of each elemental arch, derived from the geometrical decomposition of the cross-vault, comes from a sketch by Leonardo da Vinci in his Codici di Madrid, in which he clearly states his idea that *"l'arco non si romperà, se la corda dell'archi di fori non toccherà l'arco di dentro" (the arch will non crack if the chord of the outer arch will not touch the inner arch).*

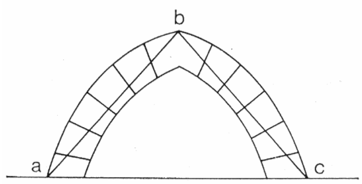

Fig. 5 Leonardo da Vinci: equilibrium condition of an arch.

The forerunner of the XVIII century graphic statics principles allowing the study of rigid blocks equilibrium, he highlights that an arch will not collapse if it is possible to find any line of thrust ("a-b" and "b-c") entirely included in its shape (Fig. 5).
To interpret properly the idea of equilibrium based on a line of thrust inside the shape of an arch, the authors propose a discrete model, that is of an arch composed of a finite and arbitrary number of rigid blocks, capable to transfer compressive forces through the interfaces (mortar joints or simple contact joints).
According to J. Heyman's hypotheses about the definition of the field of admissibility of the constitutive material (Fig. 6), proposed in the 1960s and later presented in detail in his monographs published in the 1980s-1990s [3, 4]:

- stone has an infinite compressive strength;
- sliding of one stone upon another cannot occur;
- stone has no tensile strength;

the authors deduce that:

- two adjacent blocks can never interpenetrate;
- two adjacent blocks can never slide along the contact interface, because of the presence of the friction due to the above mentioned moderate compressive force, which is always present in an arch;
- two adjacent blocks can move away each other rotating around a point placed either at the intrados or at the extrados in the contact joint.

As for its nature an arch is a double-fixed structure at the springs, it is reasonable to think that every time a block rotates relatively to another pivoting on the intrados or the extrados, here a fracture occurs and the line of thrust moves obliged to pass through the only point of the section which keeps the contact. Thus, aimed at describing this aspect in the structural model, such a fractured section can be interpreted as an internal hinge constraint.

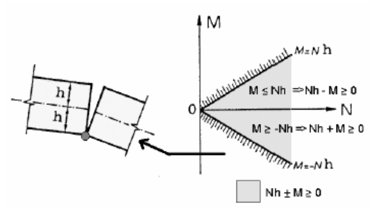

Fig. 6 J. Heyman: definition of the field of admissibility of masonry.

Therefore, it results that the only mechanism that can lead an arch to collapse is a flexural-type mechanism due to the occurring of four non aligned hinges, as previously described by C. A. Couplet (1642-1722) [5, 6] (Fig. 7).

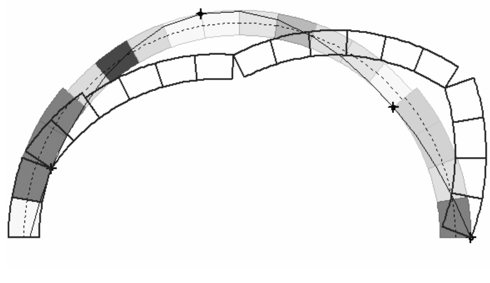

Fig. 7 Collapse mechanism of an arch and line of trust at the moment of collapse.

In order to consider all these aspects of possible mechanisms and static performances detected from the position of the line of thrust compared to the shape of the arch which, at most, can be tangent to the intrados or extrados line in three sections (three hinges arch), the contact joint (Fig. 8) is modelled by a device composed of two rods orthogonal to the interface and another tangential, in order to simulate an inner fixed-constraint.
The two rods, orthogonal to the surface, placed at the intrados and at the extrados respectively, allow the contact between the blocks and thus they are able to transfer compressive forces; in the case of tensile rods the numerical algorithm breaks them provoking a crack and forcing the line of thrust to pass through the section point opposite the fractured contact rod. Borrowing the words by J. Heyman, we can say that in masonry *"where there is a tensile force a crack occurs"*. Finally, the tangential rod simulates the friction and transfers the shear force.

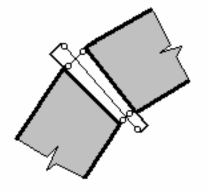

Fig. 8 Contact device between two adjacent blocks.

In such a model the aspects concerning the hypotheses of the behaviour of the material are totally supposed to be concentrated in the contact joints, considering masonry unilateral behaviour (no-tension strength) only in the definition of such joints; while the blocks of the arch, we suppose rigid, may detach only in correspondence of the joints. This is the reason why the rods arranged orthogonally to the interfaces have a rigid-cracking behaviour (Fig. 9).

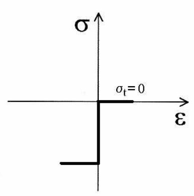

Fig. 9 The rigid-cracking behaviour of the interface rods.

4. The Computer Program "X-Vaults"

4.1 The Numerical Procedure

To evaluate the degree of stability of an arch obtained by the vault decomposition, a non linear static numerical procedure has been carried out [7]. The non-linearity of such a procedure comes from the behaviour of masonry which is in fact non-linear.

Let us consider the *i-th* arch belonging to one of the four cells when our algorithm processes it. If such an arch is composed of *n* blocks and *m* interfaces (where $m = n+1$) modelled by three rods each, when subjected to a load condition, mathematically represented by the vector *F*, the equilibrium problem can be written in the following form:

$$\begin{cases} AX = F \\ X \le 0 \end{cases} \tag{1}$$

where:

$[A]^{(3n\ x\ 3m)}$ is the geometrical configuration matrix,

$[X]^{(3m\ x\ 1)}$ is the unknown vector whose coefficients represent the forces in the interface rods,

$[F]^{(3n\ x\ 1)}$ is the known load vector.

In the system (1), the equation describes the equilibrium condition of the arch, the inequality wants all the rods orthogonal to the interface to be compressed; moreover it translates in a mathematical formula the non-linear behaviour of the constitutive material.
Leaving aside, for a moment, the sign condition imposed by the inequality, the problem comes back to the study of a system of linear algebraic equations whose solution could easily be obtained by the inverse of the matrix *A*:

$$X = A^{-1} F \tag{2}$$

in case the matrix *A* is square. It is well known that, if such a matrix had dimension $[A]^{(n\ x\ n)}$ and $\det[A] \neq 0$, only one inverse would exist, the Cayley inverse, named A^{-1}, such that:

$$AA^{-1} = A^{-1} A = I_n \tag{3}$$

where I_n is the unit matrix of order *n*.
Unfortunately, it is not possible to use this short-cut in this case, because the arch, let us repeat, is a structure with a degree of statically indeterminacy equal to three. Thus the matrix *A* is always a rectangular matrix: the number of its columns is equal to the number of its rows plus three.
This fact prevents the direct inversion of the matrix and highlights the possibility of ∞^3 solutions of the equilibrium problem and that it would be possible to choose one capable to satisfy, at most, three linearly independent and compatible conditions.
However, in the case of a unilaterally constrained structure, whose behaviour is also governed by the sign conditions of vector *X* coefficients, the solution might even not exist and, anyway, it is necessary to use other tools to find it.
You can rely on R. Penrose [8], who, in 1955, extended the notion of inverse to full rank rectangular matrices and to singular matrices or without Cayley inverse. In such

cases, given $[A]^{(r \times c)}$, being $c > r$, it is possible to define a generalized inverse named A^*.

Thus, assuming A^* the generalized inverse of A, any compatible system of the kind $AX = F$ allows a solution $X = A^* F$ if and only if the condition $AA^* F = F$ is true.

With regard to the solution of compatible linear systems the following theorem can be stated: *given the compatible system* $AX = F$ *the whole system of solutions is expressed by:*

$$X = A^* F + (I - A^* A)M \tag{4}$$

where A^* *is any inverse able to satisfy the condition* $AA^* A = A$ *and* M *an arbitrary vector of appropriate dimension.*

The general solution of the problem, written in Eq. (4), can also be rewritten in the simplified form shown in Eq. (5):

$$X = X_o + X_N \tag{5}$$

which highlights that it is the result of the sum of the two vectors:

$$X_o = A^* F$$

$$X_N = (I - A^* A)M = C\, M \tag{6}$$

where X_o represents an initial solution which is given by the use of the generalized inverse $A^* = A^T (A A^T)^{-1}$ of the matrix A we can choose, among the possible generalized inverses, following a criterion of a minimal norm; X_N represents the particular solution of the associated homogeneous system $AX = 0$ which must suitably be defined observing the sign condition of X.

Eq. (4) or equally Eq. (5), clearly highlight the unknown vector X, the solution to the static problem, cannot be obtained until the so far unknown vector M has been obtained. The vector M must be defined considering that its role is to fulfil the sign condition on the vector X.

Therefore, writing again Eq. (5) partitioning all the vectors and the matrices so as to distinguish the acceptable compressive forces (subscript c) from the unacceptable tensile ones (subscript t), we derive the following matrix expression:

$$\left[\frac{X_c}{0}\right] = \left[\frac{X_{oc}}{X_{ot}}\right] + \left[\begin{array}{c|c} C_c & B \\ \hline B^T & C_t \end{array}\right] \left[\frac{0}{M_t}\right] \tag{7}$$

which, after its solution, provides the coefficients of the sub-vector M_t:

$$M_t = -C_t^{-1} X_{ot} \tag{8}$$

and consequently the vector M:

$$M = \left[\frac{0}{M_t}\right] \tag{9}$$

From a structural point of view, observing the physical problem of the arch which, subjected to certain loads, fractures in any joints, the authors like to describe the coefficients of the vector M as *wooden wedges* driven into the joints to fill the cavity of the fracture in order to restore the continuity of the structure. In that sense, such a vector can be named *impressed distortions* vector, in Civil Engineering [9].

Once reached the vector M, that is it. Indeed, Eq. (7) allows to reach immediately the sub-vector X_c of X whose coefficients are only compressive forces:

$$X_c = X_{oc} - BC_t^{-1} X_{ot} \tag{10}$$

and consequently the whole vector X:

$$X = \left[\frac{X_c}{0}\right] \tag{11}$$

whose coefficients are all negative, or at most zeroes.

4.2 The Software Flow-Chart

The system (1) represents the physical problem of a unilaterally constrained structure; this is the reason why the initial solution X_o, referred to a bilateral constrained problem, always exists and it is unique, whereas the final solution given from Eq. (5) might also not exist.

In general, three distinct situations may occur:

- after the processing of the structure, the algorithm highlights the coefficients of the vector X_o are all negative;
- after finding the initial solution vector X_o and highlighting any positive coefficients, the algorithm turns them into zero through an iterative method which, at most, can employ *m-n* steps;
- after finding the initial solution vector X_o and highlighting all the positive coefficients, using the *m-n* steps, the algorithm still continues to detect the presence of other positive coefficients.

The first case corresponds to an arch whose interface devices are all compressed and whose line of thrust is all inside its shape, so that it is not necessary to evaluate the vector X_N, designed to correct positive forces representing tensions.

The second case corresponds to an arch whose interface devices, after a certain number of iterations, no more highlight positive forces and so the vector X_o has many coefficients zeroes as the number of the used iterations.

At last, the third case happens when the algorithm does not converge and the processed arch is unstable.

The numeric technique carried out to implement the computational code “X-Vaults” [10] can be summarized as follows:

1. compute the initial solution X_o and put $X = X_o$;

2. if X does not highlight tensile forces: end. The arch is in equilibrium with the allocated loads;
3. otherwise if X highlights one or more tensile forces, search for the highest tensile force to turn it into zero in the present step *s*;
4. to do that, extract from the initial solution X_o the sub-vector X_{os}, whose size is *s*, filled with all the coefficients greater than zero which do not satisfy the sign condition;
5. consider the sub-matrix C_S of C, square of size *s*, and evaluate the sub-vector $M_S = -C_S^{-1} X_{os}$;
6. build the vector *M* through the *s* known values of M_S, different from zero;
7. update the solution considering the relationship expressed in Eq. (5): $X = X_o + X_N$;
8. if the new solution, which contains *s* zero coefficients, shows new coefficients which do not satisfy the sign condition, check the value of *s*:
    8.1) if $s < (m-n)$ iterate the process increasing the counter *s*: $s = s+1$; go back to step 3;
    8.2) otherwise the algorithm has not reached the convergence and the arch is unstable: end;
9. otherwise if the new solution satisfies the sign condition the algorithm has reached the convergence and the arch is stable: end.

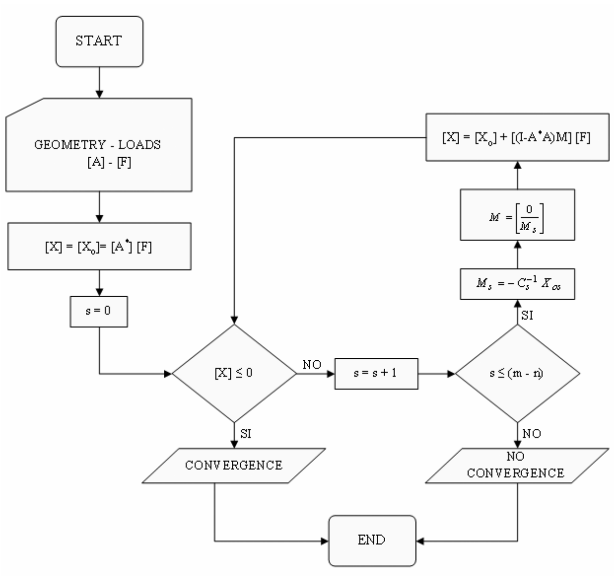

Fig. 10 Simplified flow-chart of the numerical procedure.

The flow-chart represented in Fig. 10 shows that the proposed iterative method can perform, at most, a number of steps equal to the degree of statically indeterminacy of the structure, a number which can be detected from the difference between the number of columns of *A* and the number of its rows. In case of an arch located in a two-dimensional world, such a number is equal to 3.

The software, which implements the calculation algorithm which the authors like to call, with an incorrect name, "generalized inverse method" is equipped with a friendly graphic interface and allows the user to define quickly both the geometry of the vault and its load conditions (its weight and the upper loads) through text boxes in the program windows dedicated to the data input. Software engineering [11] also allows the user a quick visualization and interpretation of the numerical results, also graphically shown by drawing the line of thrust on the shape of the arches and filling the blocks with color green, yellow and red depending on whether the line of thrust is inside, tangential or outside the blocks themselves.
The program draws the model of the structure on suitable graphic windows, some of which show the arches derived from the decomposition of the vault which the algorithm analyses (elemental arches, diagonal arches and transverse arches), both in the plane and in the elevation; instead others show an isometric overview of the vault and of the pillars.

5. A Simple Case Study

The opportunity to apply our software X-Vaults / SVM to a real case offered in 2007. At that time it was required to study the masonry structure of St. Antimo Abbey in Montalcino (SI, Italy) to understand how its static behaviour had changed as a result of the nineteenth century restoration.

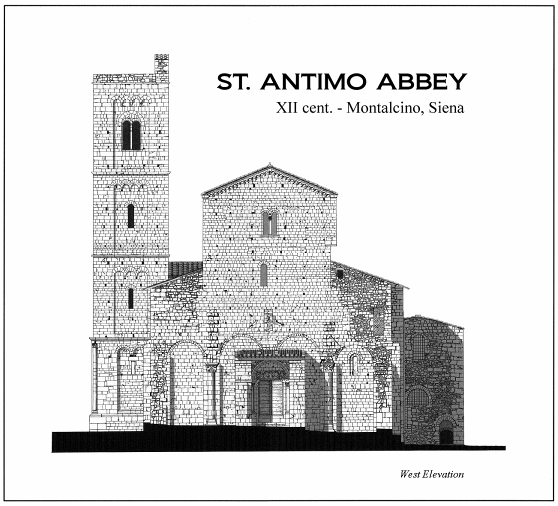

Fig. 11 St. Antimo Abbey (SI - Italy).

Built in the XII century on an earlier church, which the tradition wants to be founded by Charlemagne, the church represents an important historical example of Italian Romanesque architecture. The building has a rectangular plan, with a semicircular apse. The interior is divided into three aisles, so that the two side aisles serve as peripheral corridors round the apse too (ambulatory). The nave has a wooden ceiling with trusses at sight, while the aisles are composed of spans and are covered with cross-vaults arranged in sequence, set on tall columns. Such vaults are merged through the interposition of transverse arches set lower than the vault, and provide a base for the upper floor. The cross section of the church is typically basilican, with the nave taller than the aisles to provide high windows for the penetration of the light inside.
Avoiding to dwell too much on a detailed even if necessary historical presentation of the events that have occurred in the long run, we will neglect the preliminary investigations about the geometric survey and the study of the materials and we will not deal with the evaluation of the loads acting on the building either, even if we have carried out accurate research about this. In fact we are convinced that this is beyond the purpose of what we are discussing in this article.

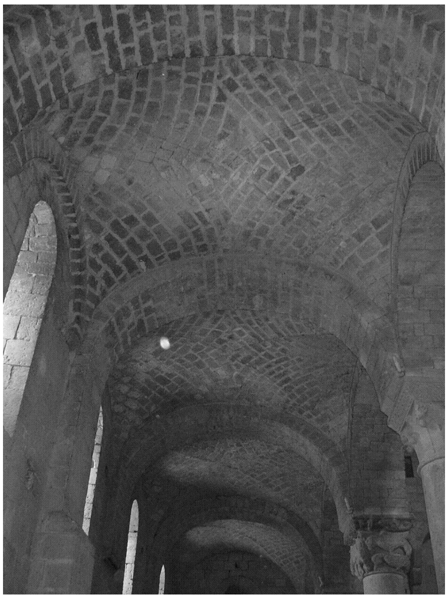

Fig. 12 Cross-vaults in the northern aisle.

So, let us focus on a cross-vault in the northern aisle: let us consider especially the vault covering the third span, which has been assumed as a *type-vault*. The vault covers a rectangular area of 2.94 x 3.14 metres, and it is obtained by a generating round arch whose span measures 2.94 metres. The structure is made of blocks of travertine and its thickness is about 0.28 metres. It is set on the northern thick outside wall and on two southern columns, which mark the spans and are 0.60 metres wide and 5.48 metres tall.
The transverse arches are round arches too, they have the same geometric features and are built with the same material of the vault (Fig. 12).

The type-vault was studied both to evaluate its degree of stability and quantify the horizontal and vertical thrusts transferred to the columns and to the wall. Thus, it was possible to verify the safety and the overturning of the pillars and the computation of the pressures both at the foundations level and on the soil too.
The analyses, obtained both by a computational FEM software and by the X-Vaults / SVM software, highlight that the solution of the equilibrium problem obtained by the first method agrees with the one of the second, but the latter is more advantageous because it provides a more targeted knowledge of the behaviour of the structure which is closer to the reality.

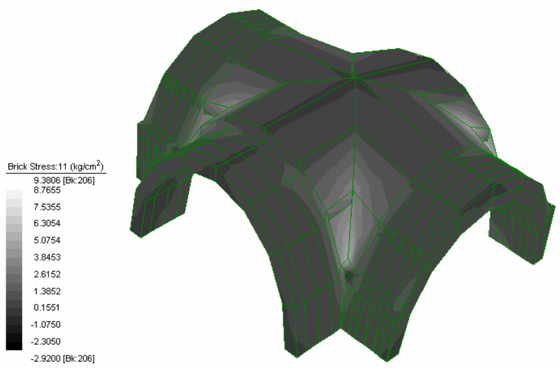

Fig. 13 FEM analysis of the cross-vault.

The structural solution carried out by the use of the FEM algorithm through a three-dimensional continuum model (Fig. 13) clearly highlights that the weakest parts of the vault are around the crossing of the cells: the diagonal arches. You can realize that, reading the stresses and observing that tensile stresses are concentrated just there. The Fig. 13 itself also shows, vice versa, that the highest compressive stresses are located in a small range of the cross-shaped extrados surface, including the keystone line of the two orthogonal barrel vaults.
The structural solution carried out by the use of the X-Vaults / SVM algorithm, shows the above mentioned critic points, but they are explained in a different manner. The model of the structure is different too; now it is designed as an assemblage of plane arched shape structures.
The Fig. 14 and the Fig. 15 clearly show that the highest compressive forces are concentrated both in a small range of the cross-shaped extrados surface and at the keystone

section of the diagonal arches. These results are coherent with the previous ones, but in this case you can also note that on the correspondent intrados surfaces there is a lack of tension because the line of thrust goes through the extrados points.

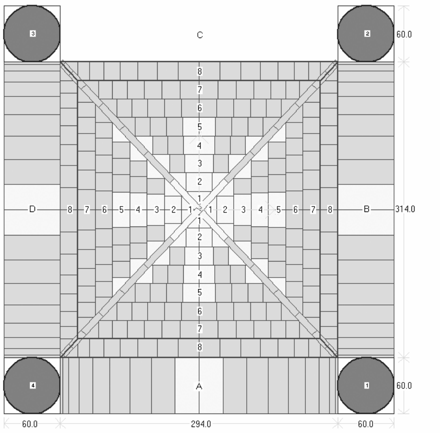

Fig. 14 Analysis of the cross-vault by X-Vaults: plane overview.

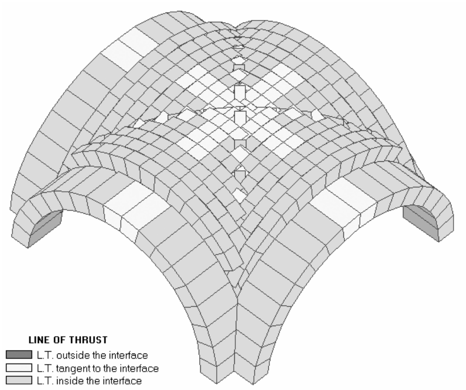

Fig. 15 Analysis of the cross-vault by X-Vaults: isometric overview.

It highlights the tendency of the structure to develop a wedge-shaped crack towards the interior of the aisle, arranged along a cross-shaped line around the keystone, but which cannot open. Looking at the elevation of the elemental arches n. 1-2-3-4 (Figs. 16-19), you can well observe how the line of thrust is tangent to the extrados at the keystone (showing the tendency to develop an intrados crack) and tangent to the intrados at the two spring sections (showing the tendency to develop an extrados crack). Thus, also making use of this kind of approach, the result is that the crossings of the barrel vaults are the more dangerous zones.

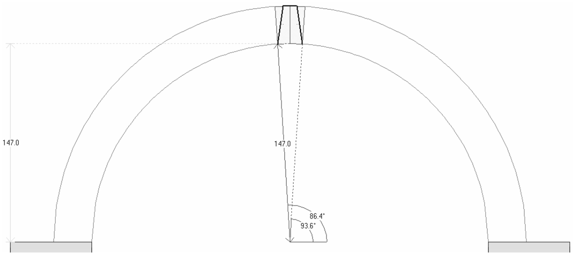

Fig. 16 Elemental arch n°1.

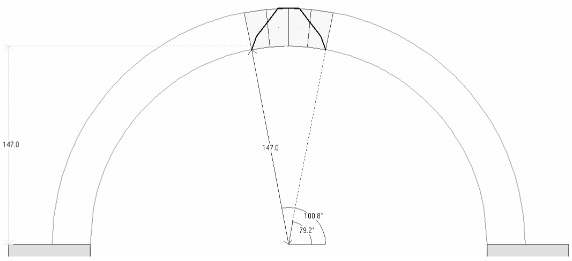

Fig. 17 Elemental arch n°2.

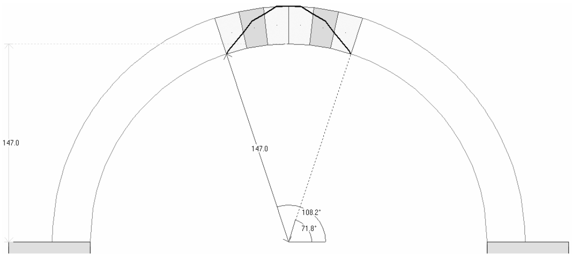

Fig. 18 Elemental arch n°3.

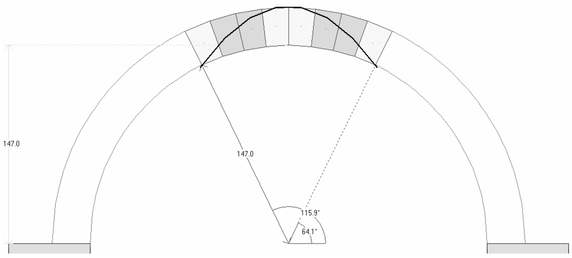

Fig. 19 Elemental arch n°4.

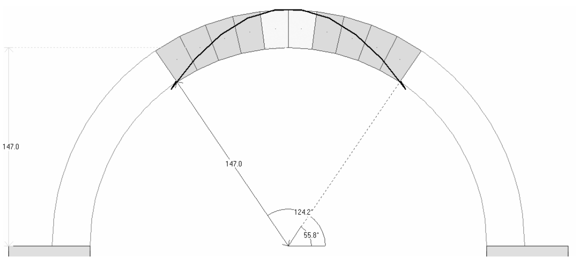

Fig. 20 Elemental arch n°5.

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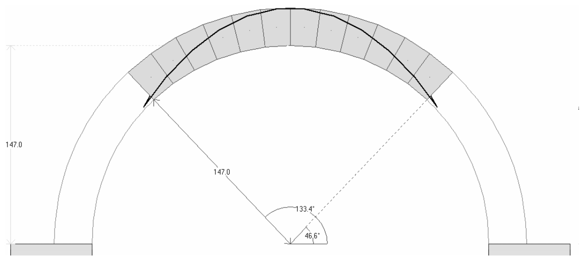

Fig. 21 Elemental arch n°6.

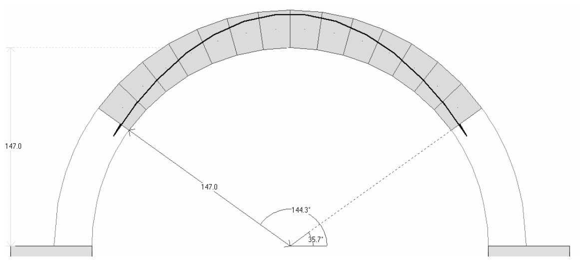

Fig. 22 Elemental arch n°7.

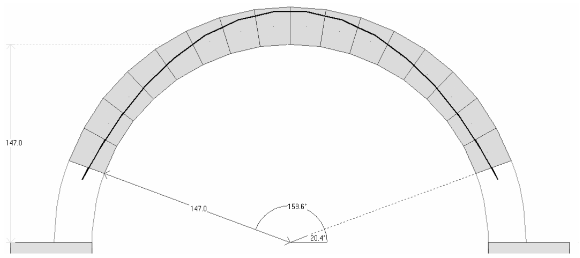

Fig. 23 Elemental arch n°8.

So, now more than ever, the words by J. Heyman *"where there is a tensile force a crack occurs"*, previously quoted, are meaningful because they can be re-proposed in the form *"where the FEM method shows tensile stresses, our algorithm highlights the tendency of the structure to develop a crack and consequently the tensile forces must be zeroes there"*.

Comparing the FEM solution with the ours, it is possible to clear an important aspect of the proposed algorithm: the vector X_o, which represents the initial solution, corresponds, in a certain sense, to the results provided by FEM. Such a solution, indeed, can also be properly defined the linear-elastic solution to the equilibrium problem, if you assume the strain of the structure is uniform and unitary concentred in the joints. The Figs. 24 and 25, referred to the analysis of one of the two diagonal arches, clearly show this concept.

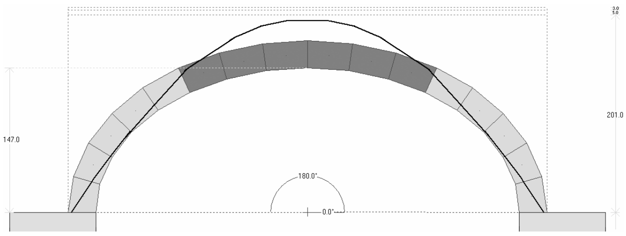

Fig. 24 Diagonal arch: initial solution X_0.

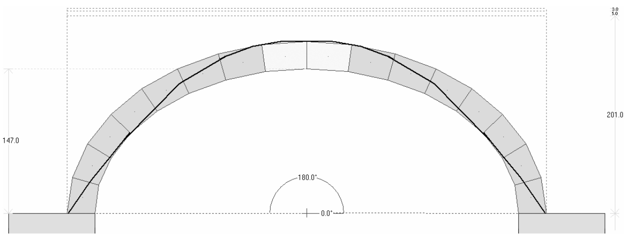

Fig. 25 Diagonal arch: final solution X.

The initial solution shows a line of thrust out of the joints of the central blocks of the arch: it means the vector X_o is filled with coefficients, some of which represent tensile forces in the intrados rods. The final solution, indeed, shows how the algorithm has been able to move the line of thrust towards the interior of the shape of the arch turning the coefficients of the tensile rods into zero, through the introduction of suitable terms of distortions (Fig. 25).

6. Conclusions

In this paper an original algorithm of structural analysis of masonry cross-vaults has been presented.
The aim to write a software by it has been to provide a simple and reliable computational tool as well as to get a deep level of knowledge of a masonry structure, in conformity with the demands of the Italian present regulations for constructions [13].
When it was translated into a software, named X-Vaults, the result of our scientific research in Civil Engineering, an Italian software-house, AEDES Software (San Miniato, Pisa, Italy) noticed it and decided to market it as the SVM (*Sistemi Voltati in Muratura*, that is *Masonry Vaulted Systems*). From 2003 to today this software has been purchased by engineers and architects, as well as by offices of Superintendents, of Civil Engineers, by Universities and by other Italian research institutions. The authors believe that the results of research should be put at everyone's disposal and agree with the Aedes slogan: *from scientific research to technique.*

Acknowledgments

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References

[1] G. A. Breymann, Archi – Volte – Cupole (from: Trattato di Costruzioni Civili, 1885), Roma: Librerie Dedalo, 2003.
[2] E. Mèry, "Mémoire sur l'équilibre des voutes", Annales des Ponts et Chausséss,, 1840, pp. 50-70.
[3] J. Heyman, The Masonry Arch, Chichester, West Sussex, England: Hellis Horwood Ltd., 1982.
[4] J. Heyman, The Stone Skeleton, Cambridge: Cambridge University Press, 1995.
[5] C. A. Couplet, De la poussée des voutes, Paris: Académie Royale des Sciences, 1731.
[6] C. A. Couplet, Seconda partie de l'examen de la poussée des voutes, Paris: Académie Royale des Sciences, 1732.
[7] S. Galassi, M. Paradiso and G. Tempesta, "A Numerical Method for No-tension Analysis of Masonry Arches", in Proc. of IV International Conference on Arch Bridges, 2004, Vol. 1, pp. 312-321.
[8] R. Penrose, "A generalized inverse for matrices", in Proc. Cambridge Philos. Soc., 1955, pp. 406-413.
[9] G. Colonnetti, Scienza delle Costruzioni, Torino: Edizioni Scientifiche Einaudi, 1955.
[10] S. Galassi, "X-Vaults: un software per lo studio della stabilità di sistemi voltati in muratura", Degree thesis, Department of Constructions, University of Florence, Florence, Italy, 2003.
[11] M. Paradiso, G. Tempesta, S. Galassi, F. Pugi, Sistemi voltati in muratura. Teoria e applicazioni, Roma: Edizioni Dei, 2007.
[12] R. Bartolini, "L'Abbazia di Sant'Antimo a Montalcino (SI). Analisi strutturale e studio del comportamento statico in seguito ai restauri ottocenteschi", Degree thesis, Department of Constructions, University of Florence, Florence, Italy, 2007.
[13] Ministero delle Infrastrutture, "Decreto Ministeriale (infrastrutture) 14 gennaio 2008. Approvazione delle nuove norme tecniche per le costruzioni", G.U. n. 29 del 4 febbraio 2008.

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