on the Controllability of a Class of
Discrete Distributed Systems

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Abstract

We consider a class of linear discrete-time systems controlled by a continuous time input. Given a desired final state $x_d$, we investigate the optimal control which steers the system, with a minimal cost, from an initial state $x_0$ to $x_d$. We consider both discrete distributed systems and finite dimensional ones. We use a method similar to the Hilbert Uniqueness Method (HUM) to determine the control and the Galerkin method to approximate it, we also give an example to illustrate our approach.

Keywords: Discrete linear systems, Hilbert Uniqueness Method, Optimal Control, Galerkin Method.

1 Introduction

This paper is devoted to the study of the controllability problem corresponding to the discrete-time varying distributed systems described by

\[
\begin{align*}
x_{i+1} &= \phi x_i + \int_{t_i}^{t_{i+1}} B_i(\theta) u(\theta) d\theta, \\
x_0 & \text{ given in } X
\end{align*}
\]

for $i = 0, \ldots, N - 1$, where $x_i \in X$, $u \in L^2(0,T,U)$, $\phi \in L(X)$, $B_i(\theta) \in L(U,X)$, $(X, \|\|)$ and $(U, \|\|)$ are Hilbert spaces and $(t_i)_{i}$ is a subdivision of the interval $[0,T]$ such that $t_0 = 0$ and $t_N = T$. Moreover, we suppose that the applications $\theta \to B_i(\theta)$, $i = 0, \ldots, N - 1$ are continuous.

In other words, given a desired final state $x_d$, we investigate the optimal control which steers the system $(S)$ from $x_0$ to $x_d$ with a minimal cost $J(u) = \|u\|$

As an example of systems described by $(S)$, we consider the linear continuous system given by

\[
x(t) = S(t)x_0 + \int_0^t S(t-r)Bu(r)dr, \quad t \geq 0 \quad (1)
\]

where $S(t)$ is a strongly continuous semi group on the Hilbert space $X$ and $B \in L(U,X)$. In order to make the system accessible by a computer we proceed to a sampling of time (see for example \cite{8,12,13}), this means, we put

\[
[0,T] = \bigcup_{i=0}^{N-1} [t_i,t_{i+1}]
\]

where

\[
\begin{align*}
t_0 &= 0 \\
t_{i+1} &= t_i + \delta,
\end{align*}
\]

with $\delta = \frac{T}{N}$ and $N \in \mathbb{N}^*$.

If we take $x_i = x(t_i)$ then
In many works (see [6, 8, 13]) and under the hypothesis system (S) leads to the difference equation

\[ x_{i+1} = x(t_{i+1}) \]

\[ = S(t_{i+1})x_0 + \int_{t_i}^{t_{i+1}} S(t_{i+1} - r)Bu(r)dr \]

\[ = S(t_i + \delta)x_0 + \int_{t_i}^{t_{i+1}} S(t_i + \delta - r)Bu(r)dr \]

\[ + \int_{t_i}^{t_{i+1}} S(t_{i+1} - r)Bu(r)dr \]

\[ = S(\delta)[S(t_i)x_0 + \int_{t_i}^{t_{i+1}} S(t_i - r)Bu(r)dr] \]

\[ + \int_{t_i}^{t_{i+1}} B_i(r)u(r)dr \]

then

\[ x_{i+1} = S(\delta)x(t_i) + \int_{t_i}^{t_{i+1}} B_i(r)u(r)dr \]

and consequently

\[ x_{i+1} = \phi x_i + \int_{t_i}^{t_{i+1}} B_i(r)u(r)dr \]

which is a system described by (S).

In many works (see [6, 8, 13]) and under the hypothesis

\[ u(t) = u_i \quad \forall t \in [t_i, t_{i+1}], \]

(1) the hypothesis (2) means that, \( u(t) \) is assumed to be constant in the interval \([t_i, t_{i+1}]\), the sampling of system (S) leads to the difference equation

\[ x_{i+1} = Lx_i + Mu_i \]

where \( L = \phi \) and \( M = \int_{t_i}^{t_{i+1}} B_i(r)dr. \)

This last discrete version has been used by several authors ([5, 3, 7, 11, 15, 16]). In some situations, the control law could have fast variations during time. Consequently the hypothesis (2) becomes inappropriate, this shows the importance of our system (S).

In this chapter, we use a technique similar to the Hilbert Uniqueness Method, introduced by Lions J.L. (see [9, 10]), in order to treat the controllability problem. The section 4 contain a method for approximating the optimal control and an example that illustrate the developed results. In the section 5, we study this problem in finite dimensional case.

### 2 Preliminary results

The final state of system (S) can be written as follows

\[ x_N = \phi^N x_0 + Hu \]

where

\[ H : L^2(0, T, U) \rightarrow X \]

\[ u \mapsto \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi^{N-j} B_{j-1}(\theta)u(\theta)d\theta. \]

\[ (3) \]

**Definition 2.1** We say that (S) is weakly controllable on \([0, \ldots, N]\) if \( \text{Im} H = X \). \( \text{Im} H \) means the range of \( H \).

**Remark 1** (S) is weakly controllable if and only if \( \text{Ker} H^* = \{0\} \).

**Lemma 1** The operator \( H \) is bounded and its adjoint operator \( H^* \) is given by , for all \( x \in X \)

\[ H^*x(\theta) = B^*_{j-1}(\theta)(\phi^*)^{N-j}x, \]

(4)

for all \( \theta \in [t_{j-1}, t_j] \) and all \( j = 1, \ldots, N \).

**Proof**

Let \( u \in L^2(0, T, U), \ x \in X \)

\[ < Hu, x > = < \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi^{N-j} B_{j-1}(\theta)u(\theta)d\theta, x > \]

\[ = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} < u(\theta), B^*_{j-1}(\theta)(\phi^*)^{N-j}x > d\theta \]

\[ = \sum_{j=1}^{N} \int_{0}^{T} < u(\theta), B^*_{j-1}(\theta)(\phi^*)^{N-j}x.X_{[t_{j-1}, t_j]}(\theta) > d\theta \]

\[ = \int_{0}^{T} < u(\theta), H^*x(\theta) > d\theta \]

hence

\[ H^*x(\theta) = \sum_{j=1}^{N} B^*_{j-1}(\theta)(\phi^*)^{N-j}x.X_{[t_{j-1}, t_j]}(\theta) \]

(5)

which implies (4).

Consider on \( X \times X \) the bilinear form given by

\[ < x, y >_F = < H^*x, H^*y >, \quad \forall x, y \in X \]

(6)

clearly, if (S) is weakly controllable, then \( < \cdot, \cdot >_F \) describes an inner product on \( X \). Let \( \| x \|_F \) be the corresponding norm and \( F \) the completion of \( X \) with respect to the norm \( \| \cdot \|_F \).
Remark 2
\[ \|x\|_F \leq \|H^*\| \|x\|, \forall x \in X. \]

In the following, we suppose that (S) is weakly controllable.

Define the operator \( \Lambda \) by
\[ \Lambda : X \to X \]
\[ x \mapsto HH^* x \]
then
\[ \text{Ker} \Lambda = \text{Ker} H^* \]
moreover
\[ \|H^* x\| = \|x\|_F, \forall x \in X \]
then, it is classical that \( \Lambda \) can be extended, in a single way by an isomorphism, denoted also \( \Lambda \), defined from \( F \) onto \( F' \) (see [10, 14]). Moreover, \( F \) is a Hilbert space with respect to the inner product
\[ \langle x, y \rangle_F = \langle \Lambda x, y \rangle_F, \forall x, y \in F \]
where \( \langle \Lambda x, y \rangle_F \) means the range of \( y \) by the operator \( \Lambda x \). From (6) we deduce that
\[ \|H^* x\| = \|x\|_F, \forall x \in X \]
hence \( H^* \) is a bounded operator from \((X, \|\cdot\|_F)\) onto \((L^2(0, T, U), \|\cdot\|)\), so it has a bounded extension, denoted \( H_* \), defined from \( F \) onto \( L^2(0, T, U) \).

Lemma 2 \( \text{Im} H \) can be identified to a subset of \( F' \).

Proof
Let \( x \in \text{Im} H \), and consider the map
\[ \varphi_x : X \to \mathbb{R} \]
\[ y \mapsto \langle x, y \rangle \]
there exists \( u \in L^2(0, T, U) \) such that \( x = Hu \), hence forall \( y \in X \) we have
\[ |\varphi_x(y)| = |\langle x, y \rangle| = |\langle Hu, y \rangle| \]
\[ = |\langle u, H^* y \rangle| \leq \|u\| \|y\|_F. \]
Consequently, \( \varphi_x \) has a bounded extension, denoted by \( \varphi_x^* \), which belongs to \( F' \). Let \( j \) be the map defined by
\[ j : \text{Im} H \to F' \]
\[ x \mapsto \varphi_x^* \]
clearly \( j \) is linear and injective.

The operator \( HH_* \) is defined from \( F \) onto \( \text{Im} H \), using lemma, (2) we can consider that \( HH_* \) is defined from \( F \) onto \( F' \).

Proposition 2.1 The operators \( \Lambda \) and \( HH_* \) are equal.

Proof
Let \( \pi \in F \) be arbitrary, we have
\[ |\langle HH_* \varphi, y \rangle_F| = |\langle HH_* \pi, y \rangle_F|, \forall y \in X \]
\[ = |\langle H_* \pi, H^* y \rangle| \]
\[ \leq \|H_* \pi\| \|H^* y\| \]
\[ \leq \|H_* \pi\| \|y\|_F \]
by density of \( X \) on \( F \), we deduce that
\[ |\langle HH_* \varphi, y \rangle_F| \leq \|H_* \varphi\| \|y\|_F, \forall y \in F \]
hence
\[ \|HH_* \varphi\|_F \leq \|H_* \varphi\| \leq \|H_* \| \|\varphi\|_F \]
which implies that \( HH_* \) is bounded. On the other hand
\[ HH_* x = H_* x = \Lambda x, \forall x \in X \]
by density of \( X \) and continuity of both \( HH_* \) and \( \Lambda \) from \( F \) onto \( F' \), we deduce that
\[ HH_* \varphi = \Lambda \varphi, \forall \varphi \in F. \]

Lemma 3 The inner product corresponding to \( \|\cdot\|_F \) is
\[ \langle x, y \rangle_F = \langle H_* x, H_* y \rangle, \forall x, y \in F \]

Proof
From (7) and Proposition 2.1, we deduce
\[ \langle x, y \rangle_F = \langle HH_* x, y \rangle_F, \forall x, y \in F \]
but
\[ \langle HH_* x, y \rangle_F = \langle HH_* x, y \rangle, \forall y \in X \]
\[ = \langle H_* x, H^* y \rangle \]
\[ = \langle H_* x, H_* y \rangle. \]
if \( y \in F \), \( \exists (y_n) \subset X \) such that \( \|y_n - y\| \to 0 \). We have,
\[ \langle HH_* x, y_n \rangle_F = \langle H_* x, H_* y_n \rangle, \forall n \in \mathbb{N} \]
when \( n \to +\infty \), we obtain
\[ \langle HH_* x, y \rangle_F = \langle H_* x, H_* y \rangle, \forall y \in F \]

Remark 3
From lemma 3, we deduce that if (S) is weakly controllable then \( \text{Ker} H_* = \{0\} \).

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3 The optimal control

We first characterize the set of all reachable states at time \(N\) from a given initial state \(x_0\).

**Proposition 3.1** The reachable set at time \(N\), from a given initial state \(x_0\), is given by

\[
R(N) = \phi^N x_0 + F'.
\]

**Proof**

If \(z \in \phi^N x_0 + F',\) then \(z - \phi^N x_0 \in F',\) hence there exists \(f \in F\) such that \(z - \phi^N x_0 = \Lambda f,\) which implies that

\[
z = \phi^N x_0 + H H_* f = \phi^N x_0 + H u
\]

where \(u = H_* f,\) thus \(z\) is reachable.

Conversely, if \(z\) is reachable, say that \(z = \phi^N x_0 + H u,\) then

\[
z - \phi^N x_0 = H u
\]

that is \(z - \phi^N x_0 \in Im H \subset F'\) hence \(z \in \phi^N x_0 + F'.\)

\(\blacksquare\)

**Theorem 3.1** If \(x_d - \phi^N x_0 \in F',\) then the control \(u^* = H_* f,\) where \(f\) is the unique solution of the algebraic equation

\[
\Lambda f = x_d - \phi^N x_0
\]

steers the system from the initial state \(x_0\) to the final state \(x_d\) at time \(N\) with a minimal cost \(J(u) = \|u\|,\) moreover \(\|u^*\| = \|f\|_F.\)

**Proof**

Let \(u^* = H_* f,\) where \(f\) verify \((8),\) \(f\) exists since \(x_d - \phi^N x_0 \in F'.\) We have,

\[
\phi^N x_0 + H u^* = \phi^N x_0 + \Lambda f = x_d
\]

hence \(u^*\) steers \((S)\) from \(x_0\) to \(x_d\) at time \(N.\) Suppose that \(v\) steers \((S)\) from \(x_0\) to \(x_d\) at time \(N,\) then

\[
\phi^N x_0 + H v = x_d = \phi^N x_0 + H u^*
\]

hence,

\[
H v = H u^*
\]

which implies that

\[
< H(v - u^*), f_n > = 0; \ \forall n
\]

where \((f_n)_n\) is a sequence, of elements in \(X,\) which converges towards \(f\) with respect to the norm \(\|\|_F.\) Consequently,

\[
< v - u^*, H_* f_n > = 0, \ \forall n
\]

or

\[
< v - u^*, H_* f_n > = 0, \ \forall n
\]

when \(n \to +\infty,\) we deduce that

\[
< v - u^*, H_* f > = 0
\]

or

\[
< v - u^*, u^* > = 0
\]

thus

\[
< v, u^* > = \|u^*\|^2
\]

which implies that

\[
\|u^*\|^2 \leq \|v\| \|u^*\|
\]

\[
\|u^*\| \leq \|v\|
\]

\(\blacksquare\)

4 A numerical approach

In order to determine the optimal control \(u^*,\) we need to resolve the algebraic equation

\[
\Lambda f = x_d - \phi^N x_0 \ \text{on} \ F'.
\]

In this section, we propose a numerical approach to approximate \(f.\) Suppose that \(x_d - \phi^N x_0 \in F'\) and that \(X\) is a separable space. Let \((w_i)_{i \geq 1}\) be a basis of \(X.\)

Equation \((9)\) is equivalent to

\[
< \Lambda f, y >_{F', F} = < x_d - \phi^N x_0, y >_{F', F}, \ \forall y \in X
\]

**Remark 4** Since the bilinear form

\[
(u, v) \to < \Lambda u, v >_{F', F}
\]

is coercive on \(F \times F\) and the map

\[
y \to < x_d - \phi^N x_0, y >_{F', F}
\]

belongs to \(F',\) one can think to apply the Galerkin method to approximate \(f.\) But this involves some difficulties because the map \(y \to < x_d - \phi^N x_0, y >_{F', F}\) is known on \(X\) but almost unknown on \(F,\) also \((u, v) \to < u, v >_F\) is known on \(X \times X\) but almost unknown on \(F \times F.\)
Equation (10) is equivalent to
\[ < f, y >_F = < x_d - \phi^N x_0, y >, \quad \forall y \in X \]  
(11)

Remark that in equation (11), the solution \( f \) belongs to \( F \) and the variable \( y \) is in \( X \). In the following, we will prove that by applying the Galerkin method to equation (11), we can construct a sequence \( (f_m) \) which converges strongly on \( F \) towards \( f \).

Let \( X_m \) be the subspace of \( X \) spanned by the vector \( w_1, w_2, \ldots, w_m \) and \( f_m \in X \), the solution of
\[ < f_m, y >_F = < x_d - \phi^N x_0, y >, \quad \forall y \in X_m \]  
(12)

Since \( \| \cdot \| \) and \( \| \cdot \|_F \) are equivalent on \( X_m \), the bilinear form \((u, v) \mapsto < u, v >_F\) is continuous and coercive on \( X_m \times X_m \), moreover, \( y \mapsto < x_d - \phi^N x_0, y > \) is bounded on \( X_m \). From the Lax-Milgram theorem, see [1, 2], we deduce that \( f_m \) exists and is unique. Using (12) we have
\[ < f_m, f_m >_F = < x_d - \phi^N x_0, f_m > \]  
(13)

Since \( x_d - \phi^N x_0 \in F' \), there exists a constant \( c \) such that
\[ | < x_d - \phi^N x_0, y >_{F', F} | \leq c \| y \|_F, \quad \forall y \in F \]  
(14)

hence,
\[ | < x_d - \phi^N x_0, y > | \leq c \| y \|_F, \quad \forall y \in X \]  
(15)

From (13) and (14), we deduce that
\[ \| f_m \|_F^2 \leq < f_m, f_m > \leq \| f_m \|_F \]  
(16)

i.e.,
\[ \| f_m \|_F \leq c, \quad \forall m. \]

Consequently, \( (f_m) \) admits a subsequence \( (f_{m'})_{m'} \), which converges weakly to a certain \( f_* \in F \), we will denote this weak convergence by
\[ f_m \rightharpoonup f_* \]  
(17)

Let \( C \) denote the set of all finite combinations of \( w_i \), \( i \geq 1 \). Suppose that \( v \in C \), then \( v \) belongs to \( X_{m'} \) for \( m' \) sufficiently large, hence
\[ < f_{m'}, v >_F = < x_d - \phi^N x_0, v >. \]

From (15), we deduce that
\[ \lim_{m' \to +\infty} < f_{m'}, v >_F = < f_*, v >_F \]
\[ = < x_d - \phi^N x_0, v >, \quad v \in C \]

let \( x \in X \), since \( C \) is dense on \( (X, \| \cdot \|) \), then there exists a sequence \( (x_n) \) such that \( \| x_n - x \|_F \to 0 \), which implies that \( \| x_n - x \|_F \to 0 \), using Remark (2). On the other hand,
\[ < f_*, x_n >_F = < x_d - \phi^N x_0, x_n >, \quad \forall n \]

when \( n \to +\infty \), we obtain
\[ < f_*, x >_F = < x_d - \phi^N x_0, x >, \quad \forall x \in X \]

hence \( f_* \) is solution of (11), by uniqueness we deduce that \( f_* = f \). Hence \( (f_m)_m \) has a subsequence \( (f_{m'})_{m'} \) which converges weakly on \( (F, \| \cdot \|_F) \) towards \( f \). Suppose that \( (f_m)_m \) doesn’t converge weakly, on \( (F, \| \cdot \|_F) \), towards \( f \), then there exists \( v \in F \) such that
\[ < f_m, v >_F \rightharpoonup f, v >_F \]

i.e.,
\[ \exists \epsilon, \forall N \exists n > N | < f_n, v >_F - < f, v >_F | > \epsilon \]

From this we deduce that, for all \( N \in \mathbb{N} \), there exists \( \varphi(N) \in N \) such that
\[ | < f_{\varphi(N)}, v >_F - < f, v >_F | > \epsilon \]

but \( (f_{\varphi(N)})_N \) is bounded on \( F \), hence \( (f_{\varphi(N)})_N \) has a subsequence \( (f_{\varphi(N')})_{N'} \), which converges weakly towards \( f \), hence
\[ < f_{\varphi(N')}, v >_F \rightharpoonup < f, v >_F \]

which contradicts (16) thus
\[ f_m \rightharpoonup f. \]

To prove that \( f_m \to f \) strongly on \( F \), we consider
\[ < f_m - f, f_m - f >_F = < f_m, f_m >_F - < f, f_m >_F - < f, f_m >_F + < f, f >_F \]

recall that
\[ < f_m, f_m >_F = < x_d - \phi^N x_0, f_m > \]

hence
\[ \lim_{m \to +\infty} < f_m, f_m >_F = < x_d - \phi^N x_0, f >_F, F \]

On the other hand,
\[ \lim_{m \to +\infty} < f_m, f > = < f, f >_F \]

\[ \lim_{m \to +\infty} < f, f_m > = < f, f >_F \]
consequently,
\[
\lim_{m \to +\infty} < f_m - f, f_m - f > = \int_{0}^{\infty} \sigma(x) \phi^{N} x_0 f >_{F,F} - < f, f >_{F} = \int_{0}^{\infty} \sigma(x) \phi^{N} x_0 \Lambda f >_{F,F} = 0
\]
thus \( f_m \to f \) strongly on \( F \).

**Remark 5** To determine \( (f_m) \), we don’t need the expression of \( H_s \) nor the completion space \( F \).

**Remark 6** The sequence of inputs \( u_n = H^* f_n \) converges strongly, on \( L^2(0,T,U) \), towards the optimal control \( u^* = H_s f \).

### 4.1 Example

Consider the system
\[
\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i
\]
where \( x(t) \in X = L^2(0,1) \), \( b_i \in X \), \( u_i \in L^2(0,T) \),
\( A = \frac{\partial^2}{\partial x^2} \) and \( D(A) = \{ x \in L^2(0,1), \frac{\partial^2 x}{\partial t^2} \in L^2(0,1), \ x(0) = x(1) = 0 \} \). \( A \) is self-adjoint and has respectively eigenvalues and eigenvectors given by \( \lambda_n = -n^2 \pi^2 \) and \( \Phi_j(t) = \sqrt{2} \sin(j \pi t), \ t \in [0,1] \) and \( j = 1,2, \ldots \).

We suppose for example that \( \int_{0}^{1} b_1(\alpha) \sin(n \alpha) d\alpha \neq 0, \ \forall n \geq 1 \), this implies that the system (17) is weakly controllable, (see [4]). If we introduce the operator \( B \)
\[
B : \mathbb{R}^m \to X \quad (u_1, \ldots, u_m) \mapsto \sum_{i=1}^{m} b_i u_i
\]
then the system (17) becomes
\[
\dot{x} = Ax + Bu.
\]  

Now, consider the discrete version of (18) obtained by a similar way as presented in the introduction of this paper,
\[
x_i+1 = \Phi x_i + \int_{t_i}^{t_{i+1}} B_i(\theta) u(\theta) d\theta
\]
where \( t_i = i \Delta, i = 0, \ldots, N \) with \( \Delta \) is a sampling of \([0,T], x_i = x(t_i), B_i(\theta) = T(t_i+1 - \theta) B, \Phi = T(\Delta)\)
where \( T(t) \) is the strongly continuous semi group, generated by \( A \), given by
\[
T(t) z = \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \Phi_n > \Phi_n^* \ , \ \forall z \in X.
\]
Since the system (18) is weakly controllable on \([0,T], \forall T > 0 \) we deduce that
\[
\forall x_d \in X, \exists u \in L^2(0,T,U) : ||x(T) - x_d|| < \epsilon
\]
which implies that
\[
\forall x_d \in X, \exists u \in L^2(0,T,U) : ||x(T) - x_d|| < \epsilon
\]
which implies that
\[
\forall x_d \in X, \exists u \in L^2(0,T,U) : ||x(T) - x_d|| < \epsilon
\]
hence (19) is also weakly controllable on \([0, T_N], \forall N \).

Since \( X \) is reflexive, then \( T^*(\Delta) \) is generated by \( A^* = A \), i.e. \( T^*(\Delta) = T(\Delta) \), which gives \( \phi^* = \phi \), and \( \phi^* = \phi^* = T(i \Delta) \).

Let’s denote \( T_N^{\Delta} = T((N-j)\Delta) \), then for any \( x \in X \), it follows from equations (3) and (4) that
\[
H H^* x = H(H^* x)
\]
\[
= \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi^{N-j} B_{j-1}(\theta) B_{j-1}(\theta)^* e^{N-j} x d\theta
\]
\[
= \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} T_{N-j}^{\Delta} T(t_j - \theta) B B^* T(t_j - \theta) T_{N-j}^{\Delta} x d\theta
\]
\[
= \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} W_j(\theta) B B^* W_j(\theta) x d\theta
\]
where \( W_j(\theta) = T((N-j)\Delta + t_j - \theta) \). On the other hand, the adjoint operator \( B^* \) of \( B \) is given by
\[
B^* : X \to \mathbb{R}^m
\]
\[
x \mapsto (b_1 x, \ldots, b_m x)
\]
If we define
\[
\alpha(n,j,\theta) = e^{-n^2 \pi^2 [t_1 - \theta + (N-j)\Delta]}
\]
\[
\phi_j^* = x, \phi_j > , x \in X, j \in \mathbb{N}
\]
then
\[
B^* T((N-j)\Delta + t_j - \theta) x
\]
\[
= (\sum_{n=1}^{\infty} \alpha(n,j,\theta) \phi^* \phi_n^* b_n + \ldots) + (\sum_{n=1}^{\infty} \alpha(n,j,\theta) \phi^* \phi_n^* b_n)
\]
thus
\[
B B^* W_j(\theta) x = \sum_{i=1}^{m} \sum_{n=1}^{\infty} e^{-n^2 \pi^2 [t_1 - \theta + (N-j)\Delta] b_i^* \phi_n^* \phi_n^* b_n}
\]
We have
\[ W_{j}(\theta)BB^{*}W_{j}(\theta)x = \sum_{k=1}^{\infty} e^{-k^{2}\pi^{2}[t_{j}-\theta+(N-j)\Delta]} < BB^{*}W_{j}(\theta)x, \Phi_{k} > \Phi_{k} \]
hence
\[ HH^{*}x = \sum_{j=1}^{N} \int_{t_{j}-1}^{t_{j}} \sum_{k=1}^{\infty} \alpha(k,j,\theta) h_{j}(x), \Phi_{k} > \Phi_{k} d\theta \]

where
\[ h_{j}(x) = \sum_{i=1}^{m} \sum_{n=1}^{\infty} \alpha(n,j,\theta) \phi_{n}^{x} \phi_{n}^{b_{i}} b_{i} \]
\[ g_{j}(x) = \sum_{i=1}^{m} \sum_{n=1}^{\infty} \alpha(n,j,\theta) \phi_{n}^{x} \phi_{n}^{b_{i}} \phi_{n}^{b_{k}} \]

Therefore
\[ < HH^{*}\Phi_{r}, \Phi_{s} > = \sum_{j=1}^{N} \int_{t_{j}-1}^{t_{j}} \sum_{r=1}^{N} \sum_{s=1}^{\infty} \alpha(r,j,\theta) \phi_{n}^{r} \phi_{n}^{b_{i}} \phi_{n}^{b_{j}} d\theta \]

Let \( \gamma_{sr} = \sum_{j=1}^{N} \int_{t_{j}-1}^{t_{j}} e^{-(s^{2}+r^{2}) \pi^{2} [t_{j}-\theta+(N-j)\Delta]} d\theta \), hence
\[ \gamma_{sr} = \sum_{j=1}^{N} \int_{t_{j}-1}^{t_{j}} e^{-(s^{2}+r^{2}) \pi^{2} [t_{j}-\theta+(N-j)\Delta]} d\theta = \frac{1}{(s^{2}+r^{2}) \pi^{2}} \left( 1 - e^{-(s^{2}+r^{2}) \pi^{2} \Delta} \right) \]

It follows from Theorem 3.1 and Remark 6 that the optimal control can be approximated by \( u_{i} = H^{*} f_{i} \)
where \( f_{i} = \sum_{l=1}^{t_{i}} z_{l} \Phi_{i} \) is the unique solution of the algebraic system
\[ < HH^{*}f_{i}, \Phi_{i} > = < x_{d} - \phi^{N} x_{0}, \Phi_{i} >, \forall i = 1,\ldots,l, \]
or equivalently
\[ A_{l} Z_{l} = X_{d} \]
where \( Z_{l} = (z_{1},\ldots,z_{l})^{t} \), \( X_{d} = (x_{d} - \phi^{N} x_{0}, \Phi_{1} >,\ldots,< x_{d} - \phi^{N} x_{0}, \Phi_{l} >)^{t} \) and \( A_{l} \) the matrix
\[ A_{l} = (\gamma_{sr} \sum_{i=1}^{m} < b_{i}, \Phi_{r} > < b_{i}, \Phi_{s} >) \mathbb{I} \]

On the other hand, from lemma 1, it follows that
\[ u_{l}(\theta) = B_{l}^{*}(\theta) (\phi^{N})^{N-j} f_{l} \]
\[ = B_{l}^{*} T(\theta) T((N-j)\Delta) f_{l} \]
\[ = B_{l}^{*} T(\theta - \theta + (N-j)\Delta) f_{l} \]
\[ = B_{l}^{*} (N\Delta - \theta) f_{l} \]

for simplicity, if we take \( m = 1 \) then,
\[ u_{l}(\theta) = b_{1}, T(N\Delta - \theta) f_{l} > \sum_{n=1}^{\infty} e^{-n^{2}\pi^{2}(N\Delta - \theta)} < f_{l}, \Phi_{n} > < b_{1}, \Phi_{n} > \]

hence, the optimal control can be approximated by for all \( \theta \in [0,T] \),
\[ u_{l}(\theta) = \sum_{n=1}^{l} e^{-n^{2}\pi^{2}(N\Delta - \theta)} < f_{l}, \Phi_{n} > < b_{1}, \Phi_{n} > . \]

\[ (20) \]

Numerical simulation : We take \( m = 1, b_{1}(t) = t^{2} + 1, N = 10, t_{i} = i\delta, \delta = 0.1, x_{0} = 0, \) then \( t_{N} = 1 \).
To have \( x_{d} \) reachable, we take \( x_{d} = Hu \) where \( u(\theta) = 1, \forall \theta \in [0,1] \), then \( x_{d} = (x_{d} - \Phi_{i} >)_{1 \leq l \leq l} \) where
\[ < x_{d}, \Phi_{i} > = \frac{b_{1} \Phi_{i} (1 - e^{-i^{2}\pi^{2} N\delta})}{\gamma_{sr}} \]

An approximation of the optimal control is then given by figure 1.
5 Finite dimensional case

In this section we take $X = \mathbb{R}^n$ and $U = \mathbb{R}$. Since $ImH$ is finite dimensional, the weak controllability of $(S)$ is equivalent to $Im H = X$, i.e., the exact controllability of $(S)$. If $(S)$ is controllable, then $\text{Ker } H^* = \{0\}$ and $\|\cdot\|_F$ is a norm on $X$ equivalent to $\|\cdot\|$, so the completion of $X$ with respect to $\|\cdot\|_F$ is $X$, i.e., $F = X$.

On the other hand, since $\Lambda = HH^*$ and $\text{Ker } \Lambda = \text{Ker } H^* = \{0\}$, then the controllability of $(S)$ implies that $\Lambda$ is an isomorphism on $X$.

Proposition 5.1 If $B_i(\theta)$, $i = 0, \ldots, N - 1$, are constant operators, say that $B_i(\theta) = B_i$, then

$$\text{Ker } H^* = \text{Ker } \begin{bmatrix} B_{N-1}^* \\ B_{N-2}^* \phi^* \\ \vdots \\ B_0^* (\phi^*)^{N-1} \end{bmatrix}$$

Proof.

If $x \in \text{Ker } H^*$, then $H^*x = 0$. From (5) it follows that

$$\sum_{j=1}^{N} B_{j-1}^* (\phi^*)^{N-j} \mathcal{X}_{t_{j-1}, t_j}(\theta)x = 0, \quad \forall \theta \in [0,T]$$

if we consider respectively $\theta \in [t_0, t_1[ \ldots, \theta \in [t_{N-1}, t_N]$, then

$$B_{j-1}^* (\phi^*)^{N-j}x = 0, \quad \forall j = 1, 2, \ldots, N$$

if we take respectively $j = 1, 2, \ldots, j = N$, then we obtain

$$B_{N-1}^* x = 0, B_{N-2}^* \phi^* x = 0, \ldots, B_0^* (\phi^*)^{N-1} x = 0,$$

which means that

$$x \in \text{Ker } \begin{bmatrix} B_{N-1}^* \\ B_{N-2}^* \phi^* \\ \vdots \\ B_0^* (\phi^*)^{N-1} \end{bmatrix}.$$  \hfill (21)

Conversely, suppose (21), then

$$B_{N-1}^* x = B_{N-2}^* \phi^* x = \ldots = B_0^* (\phi^*)^{N-1} x = 0,$$

which implies that

$$\sum_{j=1}^{N} B_{j-1}^* (\phi^*)^{N-j} \mathcal{X}_{t_{j-1}, t_j}(\theta)x = 0, \quad \forall \theta \in [0,T]$$

hence $x \in \text{ker } H^*$. \hfill \blacksquare

The operator $\Lambda$ is given by

$$\Lambda : X \rightarrow X$$

$$x \mapsto HH^* x$$

from (3) it follows that

$$HH^* x = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi^{N-j} B_{j-1}(\theta)H^* x(\theta)d\theta$$

using (4) we deduce that

$$\Lambda x = HH^* x$$

$$= \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi^{N-j} B_{j-1}(\theta)(\phi^*)^{N-j} x d\theta.$$

Finally, from theorem 3.1 we deduce the expression of the optimal control as follows.

Proposition 5.2 The control $u^* \in L^2(0,T,\mathbb{R}^p)$ given by

$$u^*(\theta) = B_{j-1}(\theta)(\phi^*)^{N-j} f, \quad \forall \theta \in [t_{j-1}, t_j[, \quad j = 1, \ldots, N$$

where $f \in \mathbb{R}^n$ is the unique solution of the algebraic equation

$$\Lambda f = x_d - \phi^N x_0$$

steers the system from the initial state $x_0$ to the final state $x_d$ at time $N$ with a minimal cost $J(u) = \|u\|$. 

Figure 1: Approximation of the optimal control

if we take respectively $j = 1, 2, \ldots, j = N$, then we obtain

$$B_{N-1}^* x = 0, B_{N-2}^* \phi^* x = 0, \ldots, B_0^* (\phi^*)^{N-1} x = 0,$$
6 Conclusion

In this paper, we have studied an optimal control problem for systems having discrete state variables and continuous-time control. We have shown that techniques similar to Hilbert Uniqueness Method can be used to resolve the problem. A numerical approach of the solution have been also developed.

References